33. *On Branched Coverings of Projective Manifolds*

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Introduction. Let M be an n-dimensional complex projective manifold. A *finite branched covering of* M is, by definition, a proper finite holomorphic mapping $\pi: X \to M$ of an irreducible normal complex space X onto M. The *ramification locus* $R_\pi = \{x \in X \mid \pi^*: \mathcal{O}_{M,\pi(x)} \to \mathcal{O}_{X,x}$ is not isomorphic$\}$ of π and the *branch locus* $B_\pi = \pi(R_\pi)$ *of* π are hypersurfaces of X and M, respectively. For a point $x \in \pi^{-1}(B_\pi)$, if $y = \pi(x)$ is a non-singular point of B_π, then x is a non-singular point of both X and $\pi^{-1}(B_\pi)$. In this case, there are coordinate systems $(z_1, \cdots, z_n)$ and $(w_1, \cdots, w_n)$ around x and y, respectively, such that π is locally given by

$$\pi: (z_1, \cdots, z_n) \longmapsto (w_1, \cdots, w_n) = (z_1, \cdots, z_{n-1}, z_n^e).$$

The positive integer e is then locally constant with respect to x. Hence, to every irreducible component D' of $\pi^{-1}(B_\pi)$, a positive integer $e = e_{D'}$ is associated and is called the *ramification index of* π *at* D'. A *covering transformation of* π is an automorphism φ of X such that $\pi\varphi = \pi$. We denote by G_π the group of all covering transformations. π is said to be *Galois* if G_π acts transitively on every fiber of π. π is said to be *abelian* if π is Galois and G_π is an abelian group.

Let $D_1, \cdots, D_s$ be irreducible hypersurfaces of M. Put $B = D_1 \cup \cdots \cup D_s$. Let $e_1, \cdots, e_s$ be positive integers greater than 1. Consider the positive divisor $D = e_1 D_1 + \cdots + e_s D_s$. A finite branched covering $\pi: X \to M$ is said to *branch at* D (resp. *at at most* D) if $B_\pi = B$ (resp. $B_\pi \subset B$) and, for every j $(1 \leqslant j \leqslant s)$, and for any irreducible component D' of $\pi^{-1}(D_j)$, the ramification index of π at D' is e_j (resp. divides e_j).

The purpose of this note is (1) to give a criterion for the existence of a finite Galois (resp. abelian) covering of M which branches at D and (2) to describe the set of all (isomorphism classes of) finite Galois (resp. abelian) coverings of M which branch at at most D. We follow the idea of Weil [4].

The detail will be given in Namba [2].

1. Abelian coverings. Let M and D be as above. Consider the additive group

$$\mathrm{Div}(M, D) = \{\hat{E} = (a_1/e_1)D_1 + \cdots + (a_s/e_s)D_s + E' \mid a_j \in Z \text{ for } 1 \leqslant j \leqslant s,\ E' \text{ is an (integral) divisor}\}$$

of rational divisors on M. $E_1, E_2 \in \mathrm{Div}(M, D)$ are said to be *linearly equivalent*, $E_1 \sim E_2$, if $E_1 - E_2$ is a principal integral divisor on M. Let

$c: H^1(M, \mathcal{O}^*) \to H^{1,1}(M, \boldsymbol{Z})$ be the map of Chern class and $j_*: H^{1,1}(M, \boldsymbol{Z}) \to H^{1,1}(M, \boldsymbol{Q})$ be the homomorphism induced by the inclusion $j: \boldsymbol{Z} \subset \boldsymbol{Q}$. Consider the subgroup

$$\begin{aligned}\operatorname{Div}_0^{\boldsymbol{Q}}(M, D) = \{\hat{E} = (a_1/e_1)D_1 + \cdots + (a_s/e_s)D_s + E' \in \operatorname{Div}(M, D) \mid c^{\boldsymbol{Q}}(\hat{E}) \\ = (a_1/e_1)j_*c([D_1]) + \cdots + (a_s/e_s)j_*c([D_s]) + j_*c([E']) \\ = 0 \in H^{1,1}(M, \boldsymbol{Q})\}\end{aligned}$$

of $\operatorname{Div}(M, D)$.

Theorem 1. *There is a bijective map $\pi \mapsto S = S(\pi)$ of the set of all (isomorphism classes of) finite abelian coverings $\pi: X \to M$ which branch at at most D, onto the set of all finite subgroups S of $\operatorname{Div}_0^{\boldsymbol{Q}}(M, D)/\sim$. The map satisfies* (1) *$G_\pi \simeq S(\pi)$ and* (2) *if $\pi_1 \leqslant \pi_2$, then $S(\pi_1) \subset S(\pi_2)$.*

Theorem 2. *There is a finite abelian covering $\pi: X \to M$ which branches at D if and only if there is a finite subgroup S of $\operatorname{Div}_0^{\boldsymbol{Q}}(M, D)/\sim$ with the following condition: for every j $(1 \leqslant j \leqslant s)$, there is $\hat{E}/\sim = \hat{E}(j)/\sim = ((a_1/e_1)D_1 + \cdots + (a_s/e_s)D_s + E')/\sim \in S$ such that $(a_j, e_j) = 1$ (coprime).*

2. Galois coverings. Let M and D be as above. Put

$$M' = M - \operatorname{Sing} B, \quad D'_j = D_j - \operatorname{Sing} B \qquad \text{for } 1 \leqslant j \leqslant s$$

and $D' = e_1D'_1 + \cdots + e_sD'_s$. Let $\{W_\alpha\}_{\alpha \in A} \cup \{W_\nu\}_{\nu \in N}$ be an open covering of M such that $B \cap W_\alpha = \phi$ for $\alpha \in A$ and $B \cap W_\nu \neq \phi$ for $\nu \in N$. An *$(r \times r)$-matrix D'-divisor* $\hat{E} = \{F_\alpha\} \cup \{F_\nu\}$ is a collection of $(r \times r)$-matrix valued meromorphic functions F_α on W_α and F_ν of $(w_1, \cdots, w_{n-1}, \sqrt[e]{w_n})$, where $(w_1, \cdots, w_n)$ is a coordinate system on W_ν such that $B \cap W_\nu = D_1 \cap W_\nu = \{w_n = 0\}$ (say), $(e = e_1)$, with the following conditions: (1) $\det(F_\alpha)$ and $\det(F_\nu)$ are not identically zero, (2) $F_\alpha F_\beta^{-1}$ etc. are holomorphic functions with never vanishing $\det(F_\alpha F_\beta^{-1})$ and (3) for any $\nu \in N$,

$$S_\nu(w_1, \cdots, w_{n-1}, \sqrt[e]{w_n}) = F_\nu(w_1, \cdots, w_{n-1}, \zeta \sqrt[e]{w_n}) F_\nu(w_1, \cdots, w_{n-1}, \sqrt[e]{w_n})^{-1}$$

is a holomorphic function with never vanishing $\det(S_\nu)$, where $\zeta = \exp 2\pi\sqrt{-1}/e$. Then the collection

$$\{F_\alpha F_\beta^{-1}\}_{\alpha, \beta \in A} \cup \{F_\alpha F_\nu^{-1}\}_{\alpha \in A, \nu \in N} \cup \{F_\nu F_\alpha^{-1}\} \cup \{F_\mu F_\nu^{-1}\}_{\mu, \nu \in N}$$

has a vector bundle like property (though $F_\mu F_\nu^{-1}$ is a holomorphic function of $(w_1, \cdots, w_{n-1}, \sqrt[e]{w_n})$). We denote this collection by $[\hat{E}]$ and call it the *D'-vector bundle* (*of rank r*) *associated with* $\hat{E}$. In a similar way, we can define generally a *D'-vector bundle* (*of rank r*).

Definition 1. A D'-vector bundle V is said to be *unitary flat* if there are a matrix D'-divisor $\hat{E}$ and a matrix meromorphic 1-form η on M with the following conditions: (1) $V = [\hat{E}]$, (2) $\hat{E}\eta\hat{E}^{-1} - (d\hat{E})\hat{E}^{-1}$ is holomorphic, (3) $d\eta + \eta \wedge \eta = 0$ and (4) the *period representation* $R_\eta: \pi_1(M - B, p_0) \to GL(r, \boldsymbol{C})$ (p_0: a fixed point) is equivalent to a unitary representation, where

$$R_\eta(\gamma) = {}_e\!\int_\gamma \eta \qquad \text{for } \gamma \in \pi_1(M - B, p_0),$$

is the analytic continuation along γ of the solution of the differential equation $dZ = Z\eta$ with the initial condition $Z(p_0) = 1$.

Note that, if B is empty, a unitary flat D'-vector bundle is nothing but a (usual) unitary flat vector bundle.

Let $UFV(M', D', r)$ be the set of all (isomorphism classes of) unitary flat D'-vector bundles of rank r. The disjoint union

$$UFV(M', D') = \bigcup_{r=1}^{\infty} UFV(M', D', r)$$

forms an associative, distributive, commutative, symmetric algebraic system (called a *Tannaka system*) with respect to direct sum, tensor product and the $*$-operation, where $*: V \mapsto {}^tV^{-1}$. An element V of $UFV(M', D')$ is said to be *irreducible* if V can not be written as a direct sum of two elements of $UFV(M', D')$. Every element V of $UFV(M', D')$ can be uniquely written as a direct sum $V = V_1 \oplus \cdots \oplus V_t$ of irreducible elements V_k $(1 \leqslant k \leqslant t)$, which are called *irreducible components of* V. A subsystem S of $UFV(M', D')$ is called a *module of* $UFV(M', D')$ if, for any $V \in S$, every irreducible component of V also belongs to S. A *representation* Ψ *of a module* S is a map

$$V \in S \longmapsto \Psi(V) \in U(r) \qquad \text{(the unitary group with } r = \mathrm{rank}(V)),$$

which is (quasi-)compatible with direct sum, tensor product and the $*$-operation, (see Tannaka [3]). The set $G(S)$ of all representations of S forms a group in a natural way.

Definition 2. A module S is said to be *finite* if (1) S is generated by finite elements and (2) $G(S)$ is a finite group.

Theorem 3. *There is a bijective map* $\pi \mapsto S = S(\pi)$ *of the set of all (isomorphism classes of) finite Galois coverings* $\pi: X \to M$ *which branch at at most* D, *onto the set of all finite modules* S *of* $UFV(M', D')$. *The map satisfies* (1) $G_\pi \simeq G(S(\pi))$ *and* (2) *if* $\pi_1 \leqslant \pi_2$, *then* $S(\pi_1) \subset S(\pi_2)$.

Theorem 4. *There is a finite Galois covering* $\pi: X \to M$ *which branches at* D *if and only if there is a finite module* S *of* $UFV(M', D')$ *with the following condition: for every* j $(1 \leqslant j \leqslant s)$, *there is* $V = V(j) \in S$ *such that there are a matrix* D'-*divisor* $\hat{E}$ *with* $V = [\hat{E}]$ *and a matrix meromorphic* 1-*form* η *on* M, *which satisfy the conditions in Definition* 1 *and such that* $R_\eta(\gamma_j)$ *has the order* e_j, *where* γ_j *is a closed curve in* $M - B$ *which rounds* D'_j *once counterclockwisely.*

3. Kato's theorem. Theorem 4 is not easy to handle. Recently, Kato [1] obtained a nice sufficient condition in a special case:

Theorem (Kato). *Let* D_j $(1 \leqslant j \leqslant s)$ *be lines on* $\boldsymbol{P}^2$. *Put*

$$\triangle = \{p \in B = D_1 \cup \cdots \cup D_s \mid m_p(B) \geqslant 3\}.$$

($m_p(B)$ *is the multiplicity of* B *at* p.) *Suppose that* $D_j \cap \triangle \neq \phi$ *for every* j. *Then, for any integers* $e_1, \cdots, e_s$ *greater than* 1, *there is a finite Galois covering* $\pi: X \to \boldsymbol{P}^2$ *which branches at* $D = e_1D_1 + \cdots + e_sD_s$.

We can generalize this theorem as follows:

Theorem 5. *Let* M *be a projective manifold of dimension greater than* 1. *Let* $D_1, \cdots, D_s$ *be irreducible hypersurfaces of* M *and* $\wedge_1, \cdots, \wedge_t$ *be fixed component free linear pencils of* M. *Suppose* (1) *every* D_j *is a member of some* $\wedge_k$ *and* (2) *every* $\wedge_k$ *contains at least* 3 D_j*'s. Then, for any integers* $e_1, \cdots, e_s$ *greater than* 1, *there is a finite Galois covering* π: $X \to M$ *which branches at* $D = e_1D_1 + \cdots + e_sD_s$.

References

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