31. *Pluricanonical Maps of Minimal 3-Folds*

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Introduction. In this paper we study pluricanonical maps of non-singular 3-folds of general type over $\boldsymbol{C}$, provided that they have minimal models. Details will be published elsewhere.

Our main result is stated as follows.

Theorem. *Let X be a minimal 3-fold of general type with index r and let K_X denote the canonical divisor. Then the n-ple pluricanonical map $\Phi_{|nK_X|}$ is birational for $n \geqq n_0$ where*

$$n_0=9 \quad \text{if } r=1,$$
$$n_0=8 \quad \text{if } r=1 \quad \text{and if } X \text{ is } \boldsymbol{Q}\text{-factorial},$$
$$n_0=13 \quad \text{if } r=2,$$
$$n_0=4r+4 \quad \text{if } 3 \leqq r \leqq 5,$$
$$n_0=4r+3 \quad \text{if } r \geqq 6.$$

For the definition of pluricanonical maps, see section 1.

The problem of the birationality of pluricanonical maps for 3-folds has been treated by Benveniste and Matsuki [5] for minimal and non-singular 3-folds. Actually they proved that $\Phi_{|nK_X|}$ is birational for $n \geqq 8$.

When we consider the birationality problem for 3-folds admitting minimal models, we can assume that the 3-folds are minimal. It is conjectured that all 3-folds of general type have minimal models.

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§ 1. Preliminaries.

Definition 1. Let X be a normal projective variety. A Weil divisor D is said to be $\boldsymbol{Q}$-Cartier if mD is a Cartier divisor for some positive integer m. X is said to be $\boldsymbol{Q}$-factorial if any Weil divisor is $\boldsymbol{Q}$-Cartier. A $\boldsymbol{Q}$-Cartier divisor D is defined to be numerically effective or nef if $(D \cdot C) \geqq 0$ for any irreducible curve C on X.

Definition 2 (Reid [6]). Let X be a normal projective variety, and K_X the canonical divisor. We say that X has only canonical singularities, if K_X is $\boldsymbol{Q}$-Cartier and for a resolution $\mu: \tilde{X} \to X$ there is a natural morphism $\mu^* \omega_X^{[s]} \to \omega_{\tilde{X}}^{\otimes s}$ for any $s \geqq 1$. The minimum integer r such that rK_X is Cartier is called the index of X.

Definition 3 (Reid [6], [7]). Let X be a normal projective variety. X is said to be minimal or a minimal model if X has only canonical singularities and if K_X is nef. Moreover if K_X is ample,we say it is a canonical variety.

For a normal projective variety X the rational map associated with the linear system $|nK_X|$ is called the n-ple pluricanonical map. If X is a minimal model, letting $\mu: \tilde{X}\to X$ be a resolution of singularities, we have $\Phi_{|nK_{\tilde{X}}|}=\Phi_{|nK_X|}\cdot\mu$. We note that the index is independent of a choice of a minimal model for a variety of general type.

We shall use the vanishing theorem of Kawamata [2] and Viehweg [8] and the following extended version by Kawamata.

Lemma 1. *Let X be a normal projective variety with only canonical singularities. Let D be a $\boldsymbol{Q}$-Cartier Weil divisor such that $D-K_X$ is nef and $(D-K_X)^{\dim X}>0$. Then $H^i(X, \mathcal{O}_X(D))=0$ for any $i>0$, where $\mathcal{O}_X(D)$ denotes the reflexive sheaf of rank 1 associated with D.*

In the proof of theorem, we reduce the problem to the birationality of certain linear systems on surfaces and apply the following proposition.

Proposition 1 (Benveniste [1], Matsuki [5]). *Let S be a non-singular surface, R a nef and big divisor on S and m a positive integer. Assume the following conditions:*

(1) *Given any two distinct points $x, y\in S$, let $\mu: \tilde{S}\to S$ be the birational morphism obtained by blowing up S at x and y, then $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(mR)-2L_x-2L_y))\neq 0$, where $L_x=\mu^{-1}(x)$ and $L_y=\mu^{-1}(y)$.*

(2) *$m\geqq 4$ or*

(2)′ *$m=3$ and $(R^2)\geqq 2$.*

Then $\Phi_{|K_S+mR|}$ is birational.

2. Outline of the proof of the theorem. If the index $r=1$, the proof is almost the same as in Matsuki [5]. Thus we shall assume $r\geqq 2$ in the following argument.

The proof will be completed if we combine the following two propositions.

Proposition 2. *Let X be a minimal 3-fold of general type with index $r\geqq 2$. We write $P(n)$ instead of $h^0(X, \mathcal{O}_X(nK_X))$ for simplicity.*

(i) *$P(n)\neq 0$ for any $n\geqq r+2$. $P(mr)\geqq 12$ for any $m\geqq 3$.*

(ii) *$|(mr+s)K_X|$ is not composed of a pencil with dimension $\geqq 3$ where r, s, m satisfy one of the following conditions:*

(1) $r=2$. $s=0$, $m\geqq 3$ or $s=1$, $m\geqq 3$.

(2) $3\leqq r\leqq 5$. $s=0$, $m\geqq 2$ or $s=1$, $m\geqq 2$ or $s\geqq 2$, $m\geqq 2$.

(3) $r\geqq 6$. $s=0$, $m\geqq 2$ or $s=1$, $m\geqq 2$ or $s\geqq 2$, $m\geqq 1$.

Proposition 3. *Let X be a minimal 3-fold of general type with index $r\geqq 2$ and s, a, k, n integers satisfying the following conditions:*

(1) *$0\leqq s<r$,*

(2) *$|(ar+s)K_X|$ is not composed of a pencil,*

(3) *$P(n-ar-s)\neq 0$,*

(4) *$P(n-kr-s-1)\neq 0$,*

(5) *$k-a\geqq 3$,*

(6) $P(ar+s) \geqq 4$,

(7) $P((k-a)r) \geqq 9$.

Then $\Phi_{|nK_X|}$ *is a birational map.*

References

[1] Benveniste, X.: Sur les applications pluricanoniques des variétés de type très général en dimension 3 (1984) (preprint).

[2] Kawamata, Y.: A generalization of Kodaira-Ramanujam's vanishing theorem. Math. Ann., **261**, 43–46 (1982).

[3] ——: The cone of curves of algebraic varieties. Ann. of Math., **119**, 603–633 (1984).

[4] Kollár, J.: Higher direct images of dualizing sheaves (1984) (preprint).

[5] Matsuki, K.: On pluricanonical maps for 3-folds of general type (1984) (preprint).

[6] Reid, M.: Canonical 3-folds, Journées de géométrie algébrique d'Angers. ed. A. Beauville, Sijthoff and Noordhof, pp. 273–310 (1980).

[7] ——: Minimal models of canonical 3-folds. Advanced Studies in Pure Math., **1**, 131–180 (1983).

[8] Viehweg, E.: Vanishing theorems. J. reine angew. Math., **335**, 1–8 (1982).