4. A Formula of Eigen-Function Expansions I. Case of Asymptotic Trees

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§ 1. In this note we present a new method of giving eigen-function expansions on a discrete set, namely a connected graph with infinitely many vertices.

Our method is to use Poisson kernels which are defined as limits of the quotient of Green kernels having different sources. This has been successfully applied to the case of symmetric spaces and free groups ([1], [5]). By technical reason we shall restrict ourselves to the case of asymptotic trees. Details will be published elsewhere (see [2] and [3]).

Let Γ be an asymptotic tree with base point O, namely there exists a compact subgraph Γ^* of Γ containing O such that the complement $\Gamma-\Gamma^*$ consists of only disjoint trees. For $\gamma, \gamma' \in \Gamma-\Gamma^*$, we say that γ is greater than γ' and denote it by $\gamma>\gamma'$ if $\mathrm{dis}(O,\gamma')<\mathrm{dis}(O,\gamma)$ and there exists a minimal geodesic segment from γ' to γ in $\Gamma-\Gamma^*$.

The symbol $\langle\!\langle \gamma_1, \cdots, \gamma_N \rangle\!\rangle$ will denote that a sequence of vertices $\{\gamma_1, \gamma_2, \cdots, \gamma_N\}$ defines a chain of a minimal geodesic segment.

Let A be a linear difference operator on $l^2[\Gamma]$ the space of $\boldsymbol{C}$-valued square summable functions on Γ:

$$(1.1) \qquad (Au)(\gamma)=\sum_{\gamma'} a_{\gamma,\gamma'} u(\gamma')$$

for $u(\gamma) \in l^2[\Gamma]$, such that 1) $a_{\gamma,\gamma'}=0$ for $\mathrm{dis}(\gamma,\gamma')\geq 2$ and $a_{\gamma,\gamma'}\neq 0$ if γ is adjacent to γ'. 2) $a_{\gamma,\gamma'}$ is real and symmetric: $a_{\gamma,\gamma'}=a_{\gamma',\gamma}$. We assume the following condition:

$$[\mathrm{C1}] \qquad \sum_{n=1}^{\infty} \min_{\substack{\mathrm{dis}(O,\gamma)=n \\ \langle\!\langle \gamma,\gamma' \rangle\!\rangle}} \frac{1}{|a_{\gamma,\gamma'}|}=\infty .$$

Then it is well-known that A defines a self-adjoint operator on $l^2[\Gamma]$ ("Hamburger's condition"). There exists the unique Green function $G(\gamma,\gamma'|z)$ representing the resolvent $(z-A)^{-1}$ which is holomorphic in z for $\mathrm{Im}\, z\neq 0$ and satisfies

$$(1.2) \qquad \sum_{\gamma'\in\Gamma} |G(\gamma,\gamma'|z)|^2<\infty .$$

Let γ_0 be an arbitrary point of Γ. Then we have the following basic

Lemma 1. *The quotient* $G(\gamma_0,\gamma'|z)/G(\gamma_0,\gamma|z)$ *is independent of* $\gamma_0 \in \Gamma$ *provided that* $\mathrm{dis}(\gamma_0,\gamma)<\mathrm{dis}(\gamma_0,\gamma')$ *and that* γ' *is adjacent to* γ for $\gamma, \gamma' \in \Gamma-\Gamma^*$. *We shall denote this ratio by* $\alpha(\gamma,\gamma'|z)$.

In fact let Γ_n be the set of vertices $\gamma\in\Gamma$ such that $\mathrm{dis}(O,\gamma)\leqq n$. We consider the Dirichlet problem for A in Γ_n:

$$(1.3) \qquad \begin{cases}(z-A)u=v & \text{in } \mathring{\Gamma}_n \\ u|_{\partial\Gamma_n}=0 \end{cases}$$

for given $v \in l^2[\mathring{\Gamma}_n]$, where $\mathring{\Gamma}_n$ denotes the open kernel of Γ_n. We denote by $G^{(n)}(\gamma, \gamma'|z)$ and $\alpha^{(n)}(\bar{\gamma}, \gamma|z)$ the corresponding Green kernel and the quotient $G^{(n)}(O, \gamma|z)/G^{(n)}(O, \bar{\gamma}|z)$ for γ adjacent to $\bar{\gamma}$ and $\bar{\gamma}<\gamma$ respectively. Then we can prove that the following limits exist:

$$(1.4) \qquad \begin{cases} \lim_{n\to\infty} G^{(n)}(\gamma, \gamma'|z) = G(\gamma, \gamma'|z) \\ \lim_{n\to\infty} \alpha^{(n)}(\gamma, \gamma'|z) = \alpha(\gamma, \gamma'|z) \qquad \text{for } \operatorname{Im} z \neq 0 \end{cases}$$

and that the ratios

$$(1.5) \qquad G(\gamma_0, \gamma'|z)/G(\gamma_0, \gamma|z) = \alpha(\gamma, \gamma'|z)$$

are independent of $\gamma_0 \in \Gamma_n$ for $\operatorname{dis}(\gamma_0, \gamma') = \operatorname{dis}(\gamma_0, \gamma)+1$. $\alpha(\gamma, \gamma'|z)$ satisfies the fundamental relations:

$$(1.6) \qquad \alpha(\bar{\gamma}, \gamma|z) = \frac{a_{\gamma,\bar{\gamma}}}{z + a_{\gamma,\bar{\gamma}} - \sum_{\gamma'} a_{\gamma,\gamma'}\alpha(\gamma, \gamma'|z)},$$

for $\bar{\gamma}<\gamma$ and $\langle\!\langle \bar{\gamma}, \gamma \rangle\!\rangle$, where γ' runs over the set where $\gamma<\gamma'$ and $\langle\!\langle \gamma, \gamma' \rangle\!\rangle$.

By using Lemma 1, we can prove the following proposition:

Proposition 1. *Let $\xi = \langle O, \gamma_1, \cdots, \gamma_m, \cdots \rangle$ be a minimal infinite geodesic line starting from O. Let $\gamma = \gamma_n$ be an element of Γ such that $\langle\!\langle O, \gamma_1, \cdots, \gamma_m, \gamma_{m+1}, \cdots, \gamma_n \rangle\!\rangle$, $\langle O, \gamma_1, \cdots, \gamma_m \rangle \subset \xi$ but not $\langle O, \gamma_1, \cdots, \gamma_m, \gamma_{m+1} \rangle \not\subset \xi$. (We denote by $\gamma \cap \xi$ the element γ_m.) Assume that $\gamma \cap \xi \in \Gamma - \Gamma^*$. Then the limit along the geodesic segment ξ*

$$(1.7) \qquad K(\gamma, \xi|z) = \lim_{\gamma'\to\xi} \frac{G(\gamma, \gamma'|z)}{G(O, \gamma'|z)} = \frac{G(\gamma, \gamma\cap\xi|z)}{G(O, \gamma\cap\xi|z)}$$

exists and is meromorphic in z for $\operatorname{Im} z \neq 0$. When Γ is itself a tree, this is simply equal to

$$(1.8) \qquad \frac{\alpha(\gamma\cap\xi, \gamma_{m+1}|z)\cdots\alpha(\gamma_{n-1}, \gamma|z)}{\alpha(O, \gamma_1|z)\cdots\alpha(\gamma_{m-1}, \gamma\cap\xi|z)}.$$

We assume further the following assumption:

[C2] $G(\gamma, \gamma'|\lambda \pm i0)$ exists for almost all $\lambda \in \sigma(A) - \sigma_p(A)$, where $\sigma(A)$ and $\sigma_p(A)$ denote the spectrum and point spectrum of A respectively.

The boundary $\partial\Gamma$ of Γ is defined to be the set of all infinite minimal geodesic segments starting from the origin O. We denote by Ξ the union of Γ and $\partial\Gamma$ which turns out to be compact by the standard topology.

Definition. We denote by $\Xi(\gamma_0)$ for $\gamma_0 \in \Gamma - \Gamma^*$, the set of all minimal infinite geodesic lines starting from O such that $\gamma_0 \in \xi$. The Radon measure $\mu(d\xi|d\lambda)$ on $\Xi \times \boldsymbol{R}$ is defined by

$$(1.9) \qquad \int_\alpha^\beta \mu(\{\gamma\}|d\lambda) = \lim_{h\downarrow 0} \frac{h}{\pi} \int_\alpha^\beta \left| G(O, \gamma|\lambda+ih) \right|^2 d\lambda = \sum_{\alpha<\lambda_j\le\beta} |u_j(O)|^2 |u_j(\gamma)|^2$$

$$(1.10) \qquad \int_\alpha^\beta \mu(\Xi(\gamma_0)|d\lambda) = \lim_{h\downarrow 0} \frac{h}{\pi} \sum_{\gamma>\gamma_0} \int_\alpha^\beta \left| G(O, \gamma|\lambda+ih) \right|^2 d\lambda.$$

These formulae make sense from the following elementary lemma:

Lemma 2. *Let $\varphi(\lambda)$ be a function of bounded variation. Let $\{\lambda_j\}$ be the set of discontinuous points of $\varphi(\lambda)$. Then*

$$(1.11) \qquad \lim_{h\downarrow 0} \frac{h}{\pi} \int_{\alpha'}^{\beta'} \left| \int_\alpha^\beta \frac{d\varphi(\mu)}{\lambda+ih-\mu} \right|^2 d\lambda = \sum_{\alpha'<\lambda_j<\beta'} \left| \varphi(\lambda_j+0) - \varphi(\lambda_j-0) \right|^2$$

for $\alpha<\alpha'<\beta'<\beta$ *and* $\alpha', \beta' \notin \{\lambda_j\}$.

In fact we have only to apply this to the formula for the spectral kernel $\theta(\gamma, \gamma' \mid \lambda)$ of A:

$$G(\gamma, \gamma' \mid z)=\int_{-\infty}^{\infty} \frac{d\theta(\gamma, \gamma' \mid \lambda)}{z-\lambda}. \tag{1.12}$$

Now we can state the Main Theorem:

Theorem of Eigenfunction Expansions. *Under the conditions* [C1] *and* [C2]

$$\begin{aligned}(1.13)\quad & \theta(\gamma, \gamma' \mid \beta)-\theta(\gamma, \gamma' \mid \alpha)\\ &=\sum_{\alpha<\lambda_j<\beta} |u_j(\gamma)|^2 |u_j(\gamma')|^2+\int_{\partial\Gamma}\int_{\alpha}^{\beta} \mu(d\xi \mid d\lambda) K(\gamma, \xi \mid \lambda+i0) K(\gamma', \xi \mid \lambda-i0)\end{aligned}$$

$\alpha, \beta \notin \{\lambda_j\}$, *where* $\{u_j(\gamma)\}$ *denote the normalized eigen-function with eigenvalue* λ_j.

In fact

$$\begin{aligned}(1.14)\quad & \theta(\gamma, \gamma' \mid \beta)-\theta(\gamma, \gamma' \mid \alpha)\\
&=\lim_{h \downarrow 0} \frac{h}{\pi} \sum_{\gamma'' \in \Gamma} \int_{\alpha}^{\beta} d\lambda G(\gamma, \gamma'' \mid \lambda+ih) G(\gamma', \gamma'' \mid \lambda-ih)\\
&=\lim_{n \to \infty} \sum_{\gamma'' \in \Gamma_n} \lim_{h \downarrow 0} \frac{h}{\pi} \int_{\alpha}^{\beta} d\lambda G(\gamma, \gamma'' \mid \lambda+ih) G(\gamma', \gamma'' \mid \lambda-ih)\\
&\quad+\lim_{n \to \infty} \lim_{h \downarrow 0} \frac{h}{\pi} \sum_{\gamma'' \in \Gamma-\Gamma_n} \int_{\alpha}^{\beta} d\lambda G(\gamma, \gamma'' \mid \lambda+ih) G(\gamma', \gamma'' \mid \lambda-ih)\\
&=\sum_{\alpha<\lambda_j<\beta} |u_j(\gamma)|^2 |u_j(\gamma')|^2\\
&\quad+\lim_{n \to \infty} \lim_{h \downarrow 0} \frac{h}{\pi} \sum_{\gamma'' \in \Gamma-\Gamma_n} \int_{\alpha}^{\beta} d\lambda \frac{G(\gamma, \gamma'' \mid \lambda+ih)}{G(O, \gamma'' \mid \lambda+ih)} \frac{G(\gamma', \gamma'' \mid \lambda-ih)}{G(O, \gamma'' \mid \lambda-ih)}\\
&\quad\times |G(O, \gamma'' \mid \lambda+ih)|^2\\
&=\sum_{\alpha<\lambda_j<\beta} |u_j(\gamma)|^2 |u_j(\gamma')|^2\\
&\quad+\lim_{n \to \infty} \sum_{\gamma'' \in \partial\Gamma_n} \int_{\alpha}^{\beta} \int_{\Xi(\gamma'')} K(\gamma, \xi \mid \lambda+i0) K(\gamma', \xi \mid \lambda-i0) \mu(d\xi \mid d\lambda)\\
&=\sum_{\alpha<\lambda_j<\beta} |u_j(\gamma)|^2 |u_j(\gamma')|^2+\int_{\partial\Gamma} \int_{\alpha}^{\beta} K(\gamma, \xi \mid \lambda+i0) K(\gamma', \xi \mid \lambda-i0) \mu(d\xi \mid d\lambda).\end{aligned}$$

Namely for $u, v \in l^2[\Gamma]$, we have

$$\sum_{\gamma \in \Gamma} u(\gamma) \overline{v(\gamma)}=\int_{\Xi \times R} \tilde{u}(\xi \mid \lambda) \overline{\tilde{v}(\xi \mid \lambda)} \mu(d\xi \mid d\lambda) \tag{1.15}$$

$$(Au, v)=\int_{\Xi \times R} \lambda \tilde{u}(\xi \mid \lambda) \overline{\tilde{v}(\xi \mid \lambda)} \mu(d\xi \mid d\lambda) \tag{1.16}$$

where $\tilde{u}(\xi \mid \lambda)$ and $\tilde{v}(\xi \mid \lambda)$ are the generalized Fourier transforms of $u(\gamma)$ and $v(\gamma)$:

$$\begin{cases}\tilde{u}(\xi \mid \lambda)=\sum_{\gamma \in \Gamma} K(\gamma, \xi \mid \lambda) u(\gamma)\\ \tilde{v}(\xi \mid \lambda)=\sum_{\gamma \in \Gamma} K(\gamma, \xi \mid \lambda) v(\gamma).\end{cases} \tag{1.17}$$

§2. Example of periodic trees. Let Γ be a finite connected graph with base point O and $\tilde{\Gamma}$ be its universal covering graph, namely the set of all finite minimal geodesic segments starting from O. $\tilde{\Gamma}$ is a connected tree which is locally bounded. Let $\tilde{\Gamma}_0$ be the set of all closed minimal geodesic loops with base point O, so that Γ can be regarded as the quotient $\tilde{\Gamma}_0\backslash\tilde{\Gamma}$. We denote by π the projection: $\tilde{\Gamma}\to\Gamma$. We consider a linear difference operator of nearest neighbours A on $l^2(\Gamma)$ satisfying the properties (1.1). A is also regarded as a bounded linear self-adjoint difference operator on $l^2(\tilde{\Gamma})$ which is invariant with respect to the action of $\tilde{\Gamma}_0$. We denote by $W_{\tilde{\gamma}}$ for $\tilde{\gamma}\in\tilde{\Gamma}$ the inverse of the diagonal of the Green kernel $G(\tilde{\gamma},\tilde{\gamma}\,|\,z)$. Remark that $W_{\tilde{\gamma}}$ depends only on $\gamma=\pi(\tilde{\gamma})$ so we also denote it by W_γ. Then we have

Theorem 2. *The functions $W_\gamma(z)$ are characterized by the following algebraic system of equations*

$$(2.1)\qquad z-a_{\gamma,\gamma}-W_\gamma=\sum_{\langle\langle\gamma,\gamma'\rangle\rangle}\frac{1}{2}\left(-W_\gamma+\sqrt{W_\gamma^2+\frac{4a_{\gamma,\gamma'}^2W_\gamma}{W_{\gamma'}}}\right)$$

for $\gamma\in\Gamma$ and the asymptotic forms

$$(2.2)\qquad W_\gamma(z)=z+O(1)\qquad for\ |z|\gg 1.$$

Moreover for $\langle\langle\tilde{\gamma},\tilde{\gamma}'\rangle\rangle$ for $\tilde{\gamma},\tilde{\gamma}'\in\tilde{\Gamma}$ and $\operatorname{dis}(\tilde{\gamma}'',\tilde{\gamma})<\operatorname{dis}(\tilde{\gamma}'',\tilde{\gamma}')$, *the quotient* $\alpha(\tilde{\gamma},\tilde{\gamma}'\,|\,z)=\dfrac{G(\tilde{\gamma}'',\tilde{\gamma}'\,|\,z)}{G(\tilde{\gamma}'',\tilde{\gamma}\,|\,z)}=\dfrac{G(\tilde{\gamma},\tilde{\gamma}'\,|\,z)}{G(\tilde{\gamma},\tilde{\gamma}\,|\,z)}$ *is equal to*

$$(2.3)\qquad \frac{-W_\gamma+\sqrt{W_\gamma^2+4a_{\gamma,\gamma'}^2W_\gamma/W_{\gamma'}}}{2a_{\gamma,\gamma'}}.$$

For proof, see [3].

Corollary 1. *The Green function $G(\tilde{\gamma},\tilde{\gamma}'\,|\,z)$ is an algebraic function of z. So that it satisfies the conditions* (C1)–(C2).

By using (2.1), we can also prove that $G(\tilde{\gamma},\tilde{\gamma}\,|\,z)$ *has no real poles.* Hence we have

Corollary 2. *A has no point spectrum on $l^2(\tilde{\Gamma})$. A has only a finite bands of absolute continuous spectrums.*

A special case of the formula (1.14) has been given in case where $\tilde{\Gamma}$ is a free group and Γ consists of only one point. (See [1].) When Γ is equal to $\boldsymbol{Z}$, our result completely agrees with a result in [8] or more generally in [6]. As a generalization of periodic Toda lattices, one may ask the following

Question. Is the number of continuous bands of spectrums of A equal to or smaller than the number of vertices of Γ?

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