29. A Generalization of Gauss' Theorem on the Genera of Quadratic Forms*)

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Let T be a torus defined over $\boldsymbol{Q}$. As is well known, one can associate with T the class number h_T independently of matrix representation of T. (See [2] p. 119 footnote and p. 120 line 17. As for basic facts on tori, see [2], [3].) When $T=R_{K/\boldsymbol{Q}}(G_m)$, the multiplicative group $K^\times$ of an algebraic number field K viewed as an algebraic group over $\boldsymbol{Q}$, h_T coincides with the ordinary class number h_K of the field K. Consider a short exact sequence of tori over $\boldsymbol{Q}$:

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0.$$

It is natural to think of the alternating product

$$\frac{h_T}{h_{T'}h_{T''}}.$$

In his thesis Shyr considered this problem, obtained a general formula using [2], [3] and noticed, among others, that the formula is nothing but the formula of Gauss

(G) $$h_K^+ = h_K^* 2^{t-1}$$

when applied to $T=R_{K/\boldsymbol{Q}}(G_m)$, $T''=G_m$ and $T'=$the kernel of the norm map $N: T \to T''$, where $K/\boldsymbol{Q}=$a quadratic extension, $h_K^+=$the class number of K in the narrow sense, $h_K^*=$the number of classes in a genus and $t=$the number of rational primes ramified in K/Q. (See [4] and [5].)

In this note, we shall report formulas of the same type as (G) for any cyclic Kummer extension K/k and clarify the relationship between ingredients of our formula and those appearing in the classical treatment of class field theory.

So, let k be an algebraic number field of degree n_0 over $\boldsymbol{Q}$ which contains a primitive n-th root of 1 ($n \geqq 2$) and K/k be a cyclic extension of degree n. Consider tori $T_0=R_{K/k}(G_m)$, $T_0''=G_m$ over k and the exact sequence over k:

$$0 \longrightarrow T_0' \longrightarrow T_0 \longrightarrow T_0'' \longrightarrow 0$$

where T_0' is the kernel of the norm map $N: T_0 \to T_0''$. Applying $R_{k/\boldsymbol{Q}}$, we obtain the exact sequence over $\boldsymbol{Q}$:

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$$

where $T=R_{k/\boldsymbol{Q}}(T_0)=R_{K/\boldsymbol{Q}}(G_m)$, $T''=R_{k/\boldsymbol{Q}}(G_m)$ and $T'=R_{k/\boldsymbol{Q}}(T_0')$. We have $h_T=h_K$, $h_{T''}=h_k$. As for the Tamagawa numbers, we have $\tau(T)=\tau(T'')=1$ and $\tau(T')=\tau_k(T_0')=n$ since K/k is cyclic of degree n. (See [3] Corollary to

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Main Theorem and § 6, pp. 69–70.) Call λ_0 the isogeny $T_0 \to T'_0 \times T''_0$ defined over k given by $\lambda_0(x)=(x^n(Nx)^{-1}, Nx)$ and apply $R_{k/Q}$ to get an isogeny over $\boldsymbol{Q}$:

$$\lambda = R_{k/Q}(\lambda_0) : T \longrightarrow T' \times T''.$$

Then, Shyr's formula yields

$$\frac{h_K}{h_k h_{T'}} = \frac{1}{n} \frac{q(\lambda(\boldsymbol{R}))}{q(\lambda(\boldsymbol{Z}))q(\hat{\lambda}(\boldsymbol{Q}))} \prod_{p \neq \infty} q(\lambda(\boldsymbol{Z}_p)), \tag{1}$$

(see [4] p. 33, Theorem 3.1.1 or [5] p. 372, Theorem 2), where $q(\alpha)=[\mathrm{Cok}\,\alpha]/[\mathrm{Ker}\,\alpha]$ for a homomorphism α of abelian groups and various homomorphisms on the right hand side of (1) are obtained naturally from the isogeny $\lambda : T \to T' \times T''$ over $\boldsymbol{Q}$.[1] The values of q-symbols in (1) are

$$q(\hat{\lambda}(\boldsymbol{Q}))=1, \tag{2}$$

$$q(\lambda(\boldsymbol{R}))=n^{(n-1)r_2},\text{[2]} \tag{3}$$

$$q(\lambda(\boldsymbol{Z}))=n^{r_K - r_k - 1}[H^1(G, \mathfrak{o}_K^\times)]2^\rho,\text{[3]} \tag{4}$$

$$\prod_{p \neq \infty} q(\lambda(\boldsymbol{Z}_p)) = n^{n_0(n-1)} \prod_{\mathfrak{p}} e_{\mathfrak{p}}(K/k).\text{[4]} \tag{5}$$

Applying (2), (3), (4), (5) to (1), we get

$$\frac{h_K}{h_k h_{T'}} = \frac{n^{(n-2)r_2 + R_2} \prod e_{\mathfrak{p}}(K/k)}{2^\rho [H^1(G, \mathfrak{o}_K^\times)]}. \tag{6}$$

If $n \geqq 3$, since k contains a primitive n-th root of 1, k is totally imaginary, i.e. $r_1 = R_1 = 0$, hence $n_0 = 2r_2$, $nn_0 = 2R_2$, $\rho = 0$, and so

$$\frac{h_K}{h_k h_{T'}} = n^{n_0(n-1)} \frac{\prod e_{\mathfrak{p}}(K/k)}{[H^1(G, \mathfrak{o}_K^\times)]}, \qquad n \geqq 3. \tag{7}$$

If, in particular, $n = l$ a prime $\geqq 3$, then we have

$$\frac{h_K}{h_k h_{T'}} = l^{n_0(l-1)+t-e}, \qquad l \geqq 3, \tag{8}$$

where t = the number of prime ideals of k ramified for K/k and e is an integer such that $H^1(G, \mathfrak{o}_K^\times) = (\boldsymbol{Z}/l\boldsymbol{Z})^e$. On the other hand, when $n=2$, we have

$$\frac{h_K}{h_k h_{T'}} = 2^{2r_2 + t - e}, \qquad n=2, \tag{9}$$

where t, e are defined as in (8). The Gauss' formula (G) is easily seen to be a special case of (9) where $k = \boldsymbol{Q}$.

In the classical treatment of a cyclic extension K/k, one encounters the number a of ambiguous ideal classes. a is the number of ideal classes left fixed by the action of $G = \mathrm{Gal}(K/k)$. (See [1] p. 402 and p. 406 for a formula of a.) When K/k is a cyclic Kummer extension, one finds the relation:

$$2^\rho h_K = n^{(n-2)r_2 + R_2} a h_{T'}. \tag{10}$$

1) $[*]$ means the cardinality of a set $*$.

2) For k, we put $r_k = r_1 + r_2 - 1$ where r_1 is the number of real places and r_2 is the number of pairs of complex places of k. Similarly, we put $r_K = R_1 + R_2 - 1$ for K.

3) Here $G = \mathrm{Gal}(K/k)$ and $H^1(G, \mathfrak{o}_K^\times)$ = the 1st cohomology group of the G-module $\mathfrak{o}_K^\times$, the group of units of K. ρ = the number of real places of k ramified for K/k.

4) $e_{\mathfrak{p}}(K/k)$ is the ramification index of a finite prime $\mathfrak{p}$ of k for K/k.

In other words, we have

(11) $$h_K = n^{n_0(n-1)} a h_{T'}, \qquad \text{if } n \geqq 3,$$

and

(12) $$h_K = 2^{2r_2} a h_{T'}, \qquad \text{if } n = 2.$$

If, in particular, $K=\boldsymbol{Q}(\zeta)$, $k=\boldsymbol{Q}(\zeta+\zeta^{-1})$ where ζ is a primitive m-th root of 1 for an odd prime m, then since $r_2=0$, $t=e=1$, (9) yields $h_K=h_k h_{T'}$ and so, by (12), we get $a=h_k$.

References

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