28. On Semi-idempotents in Group Rings

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After Gray [1], an element $\alpha\neq 0$ of a ring R is called *semi-idempotent* if and only if α is not in the proper two-sided ideal of R generated by $\alpha^2-\alpha$, i.e. $\alpha\notin R(\alpha^2-\alpha)R$ or $R=R(\alpha^2-\alpha)R$. 0 is also counted among semi-idempotents. It is obvious that idempotent element is semi-idempotent. Throughout this note, K denotes a (commutative) field. We are concerned here with the group ring $R=KG$ over a group G. § 1 contains some propositions of general nature. In § 2 we prove a theorem for the case where G is abelian.

§ 1. **Trivial and non-trivial semi-idempotents.** In the following, we consider the group ring $R=KG$, $G\neq 1$. It is easily seen that for $k\in K$ the element $k\cdot 1\in R$ is semi-idempotent. Semi-idempotents of this form are called *trivial*, other semi-idempotents *non-trivial*. The subset $\{\sum_{g\in G}a_g g;\ \sum_{g\in G}a_g=0\}$ forms a proper two-sided ideal of R, called the *augmentation ideal* $w(R)$ of R (Passman [2]).

Proposition 1. *The group ring $R=KG$ $(G\neq 1)$ contains non-trivial semi-idempotents.*

Proof. Any element g of $G-\{1\}$ is non-trivial semi-idempotent because $g\notin w(R)$, $g^2-g\in w(R)$.

Proposition 2. *If H is a subgroup of G of finite order n, $\alpha=(\sum_{h\in H}h)+1$ is a non-trivial semi-idempotent.*

Proof. We have $\alpha^2-\alpha=(n+1)\sum_{h\in H}h$. If $n+1=0$ in K, α is idempotent. If $n+1\neq 0$ in K, we have $R(\alpha^2-\alpha)R=R(\sum_{h\in H}h)R$, so that $\alpha\in R(\alpha^2-\alpha)R$ implies $1=\alpha-\sum_{h\in H}h\in R(\alpha^2-\alpha)R$ whence $R=R(\alpha^2-\alpha)R$. Thus α is semi-idempotent.

Proposition 3. *If α is non-trivial idempotent of $R=KG$* (i.e. $\alpha\in R$, $\alpha^2=\alpha$ *and* $\alpha\notin\{0,1\}$), *$\alpha+1$ is semi-idempotent.*

Proof. Put $\beta=\alpha+1$. Then we have $\beta^2-\beta=\alpha\beta=\alpha^2+\alpha=2\alpha$. If $2=0$ in K, β is idempotent. If $2\neq 0$ in K, we have $R(\beta^2-\beta)R=R\alpha R$. Therefore $\beta\in R(\beta^2-\beta)R$ implies $\alpha+1\in R\alpha R$, $R(\beta^2-\beta)R=R$. Thus β is semi-idempotent.

§ 2. **Abelian case.** Now we consider the case where $R=KG$ is a group ring over an abelian group G. Then every ideal in R is of course two-sided.

Proposition 4. *Let $R=KG$ be a group ring over an abelian group G. If α $(\neq 0)$ is semi-idempotent but not a unit in R, then $\alpha-1$ is not a unit in R.*

Proof. Suppose $\alpha-1$ be a unit in R. Then there is an element β of

R such that $(\alpha-1)\beta=1$. Thus we would have $\alpha=(\alpha^2-\alpha)\beta\in(\alpha^2-\alpha)R$ but $(\alpha^2-\alpha)R\neq R$ because α, and hence $\alpha(\alpha-1)=\alpha^2-\alpha$ is not a unit. Thus α would not be semi-idempotent.

Proposition 5. *Let $R=KG$ be a group ring over an abelian group G. If $\alpha\in R$ is not a zero-divisor and $\alpha-1$ is not a unit in R then α is semi-idempotent.*

Proof. Suppose α be not semi-idempotent. Then $(\alpha^2-\alpha)R$ is a proper ideal of R and $\alpha\in(\alpha^2-\alpha)R$. Thus there is an element $\beta\in R$ such that $\alpha=(\alpha^2-\alpha)\beta=\alpha(\alpha-1)\beta$. As α is not a zero-divisor, we would have $1=(\alpha-1)\beta$, which would mean that $\alpha-1$ is a unit in R.

Note. It is obvious that elements of $R=KG$ of the form kg, $k\,(\neq 0)\in K$, $g\in G$ are units of R. They are called *trivial units*, other units *non-trivial*. It was proved in Passman [2] Chapter 13 that if G is a torsion free abelian group (actually G can be a group of more general type), $R=KG$ has no proper zero-divisors and all units of R are trivial. Using this, we obtain the following theorem, which is the main result of this paper.

Theorem. *Let $R=KG$ be the group ring over a torsion free abelian group G. Let $\alpha\neq 0$ be an element of R which is not a unit. Then α is semi-idempotent if and only if $\alpha-1$ is not a trivial unit.*

Proof. The only-if-part follows from Proposition 4 and the if-part from Proposition 5 and Passman's result.

Remark. The following problems remain open but seem difficult to solve.

(1) Can Proposition 5 be extended into the form: Let K be a field and $R=KG$ the group ring over any group G. If $\alpha-1$ is not a unit in R, then α is semi-idempotent?

(2) Can our Theorem be extended into the form: Let K be a field and $R=KG$ the group ring over any torsion free group G, and suppose $\alpha\,(\neq 0)\in R$ and that α is not a unit. Then α is semi-idempotent if and only if $\alpha-1$ is not of the form kg, $k\in K$, $g\in G$?

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Corrigenda to my former paper in Proc. Japan Acad, **60A**, 333–334 (1984).

p. 333 line 11 from bottom, add "or" between "$1<i$" and "$1<j$".

p. 334 line 7 from above, add "$p\geq$" before "$k\geq 2$".

p. 334 line 10 from bottom, read "$e=a\cdot 1$, $a^2=a\in R$" instead of "$e=0$ or $e=1$".

References

[1] Gray, M.: A Radical Approach to Algebra. Addison Wesley (1970).

[2] Passman, D. S.: The Algebraic Structure of Group Rings. Wiley-Interscience, New York (1977).