27. *On Removable Singularities of Certain Harmonic Maps*

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1. Statement of result. Let Ω be a domain in $\boldsymbol{R}^n$ and let (M, g) be a Riemannian manifold of dimension m. We assume that M is isometrically embedded in Euclidean space $\boldsymbol{R}^k$. The equation of harmonic maps from Ω into M is given as follows.

$$\Delta u^\alpha(x)=\sum_{i=1}^n A^\alpha_{u(x)}(D_i u(x), D_i u(x)) \qquad \alpha=1, \cdots, k \tag{1.1}$$

where $A_{u(x)}(\cdot, \cdot)$ is the second fundamental form of M at $u(x)$. This is the Euler-Lagrange equation of the energy functional

$$E(u)=\int_\Omega e(u)(x)dx \qquad \text{where } e(u)(x)=|Du(x)|^2. \tag{1.2}$$

(Hereafter, we denote $e(u)(x)$ simply as $e(u)$.)

The purpose of this article is to give a regularity result for a certain class of weak solutions of (1.1). $H^1(\Omega, \boldsymbol{R}^k)$ denotes the Sobolev space of order 1 from Ω to $\boldsymbol{R}^k$. $H^1(\Omega, M)$ is the subset of $H^1(\Omega, \boldsymbol{R}^k)$ consisting of maps having image almost everywhere in M and $L^\infty(\Omega, M)$ is defined similarly.

Definition 1.1 ([8]). A map $u \in H^1(\Omega, M) \cap L^\infty(\Omega, M)$ is called a *stationary map* if the following conditions are satisfied.

(1) For any $\eta \in C_0^\infty(\Omega, \boldsymbol{R}^k)$ we have

$$\int_\Omega \sum_{\alpha=1}^k \sum_{i=1}^n (D_i u^\alpha D_i \eta^\alpha + A^\alpha_u(D_i u, D_i u)\eta^\alpha)dx=0. \tag{1.3}$$

(Then, u is called a *weakly harmonic map*.)

(2) For each one-parameter family $\{F_t\}$ of diffeomorphisms of Ω which are equal to the identity outside a compact set of Ω and with $F_0=\text{id.}$, we have

$$(d/dt)E(u\circ F_t)|_{t=0}=0. \tag{1.4}$$

Remark 1.2. It is known that continuous harmonic maps are smooth stationary maps (see [8]).

The main result is as follows.

Theorem 1.3. *Let B be the unit ball in $\boldsymbol{R}^n$ $(n\geqq 3)$ with the center at the origin and let (M, g) be a Riemannian manifold of dimension m. Let $u \in H^1(B, M) \cap L^\infty(B, M)$ be a stationary map. Suppose that u is of class C^2 in $B-\{0\}$ and the integral $\int_B |Du|^n dx$ is finite. Then, u is extended as a smooth harmonic map from B to M.*

Remark 1.4. (1) In case $n=2$, isolated singular points are removable for each weakly harmonic map ([7, Theorem 3.6]).

(2) In the above theorem the assumption that $\int_B |Du|^n\,dx<\infty$ is necessary in general. Indeed, the result of Jäger-Kaul [4] states that the equator map u_e from B in $\boldsymbol{R}^n$ into S^n defined by $u_e(x)=(x/|x|, 0)$ is an energy minimizing map for $n\geqq 7$, that is, u_e is a stationary map with isolated singularity not satisfying $\int_B |Du_e|^n\,dx<\infty$.

Detailed proof under more general situation will be given elsewhere.

2. Growth estimate of gradient. Here we derive an estimate of the gradient Du near the singular point 0. Since u is smooth in $B-\{0\}$, the following Bochner formula holds in $B-\{0\}$.

$$\frac{1}{2}\Delta e(u)=|D^2u|^2-\sum_{i,j}\langle R^M(u_*e_i, u_*e_j)u_*e_i, u_*e_j\rangle \tag{2.1}$$

where $\{e_i\}$ is an orthonormal basis for $\boldsymbol{R}^n$, u_* is the differential of u and R^M denotes the Riemannian curvature tensor of (M, g). Thus, we have

$$\Delta|Du|\geqq -K|Du|^3 \qquad \text{in } B-\{0\}, \tag{2.2}$$

where $K=K(n, M)$. We regard $Ke(u)=b$ as a fixed function and write the above inequality as

$$\Delta f+bf\geqq 0 \qquad \text{in } B-\{0\}, \tag{2.3}$$

for $f=|Du|$. In (2.3) our case corresponds to $b\in L^{n/2}$ so we cannot apply the de Giorgi-Nash-Moser iteration method as in [2], [5]. Using the modified argument in [9] we obtain the following result.

Proposition 2.1. *Suppose that $u\in H^1(B, M)\cap L^\infty(B, M)$ is weakly harmonic and u is of class C^2 in $B-\{0\}$. There exists $\varepsilon>0$ depending only on n, M such that if $\int_{B(0,R)}|Du|^n\,dx\leqq\varepsilon$ for some $R>0$, then u satisfies the inequality*

$$|Du(x)|^2\leqq C_1|x|^{-n}\int_{B(0,2|x|)}|Du|^2\,dx\leqq C_2\varepsilon^{2/n}|x|^{-2}$$

for any $x\in B(0, R/2)-\{0\}$ where C_1, C_2 depend only on n, M.

3. Monotonicity formula. For stationary maps the following formula is known (see [6]).

Proposition 3.1 (Monotonicity formula). *Suppose that u is a stationary map from a domain Ω in $\boldsymbol{R}^n$ into a Riemannian manifold M. Then, for any $x_0\in\Omega$ and $0<\sigma<\rho<\operatorname{dist}(x_0, \partial\Omega)$,*

$$\sigma^{2-n}\int_{B(x_0,\sigma)}e(u)dx+2\int_{B(x_0,\rho)-B(x_0,\sigma)}r^{2-n}|D_r u|^2\,dx =\rho^{2-n}\int_{B(x_0,\rho)}e(u)dx \tag{3.1}$$

where $r=|x-x_0|$.

As a corollary of Proposition 3.1, we easily obtain the following lemma.

Lemma 3.2. *Suppose that u, Ω, M are given as above. Then, for any $x_0\in\Omega$ and $0<\rho<\operatorname{dist}(x_0, \partial\Omega)$, we have*

$$\int_{B(x_0,\rho)}e(u)dx\leqq 2\int_{B(x_0,\rho)}|D_\omega u|^2\,dx \tag{3.2}$$

where $|D_\omega u|^2$ *denotes the tangential energy along the sphere* $|x-x_0|=r$, *so that* $e(u)=|Du|^2=|D_r u|^2+|D_\omega u|^2$.

4. Proof of the main theorem. By the regularity theorem of [3] it is sufficient to show that u is Hölder continuous in a neighborhood of 0. We choose $R>0$ as in Proposition 2.1. For fixed $\rho\in(0, R/2)$, we may take $v\in H^1(B_\rho, \boldsymbol{R}^k)$ satisfying:

(1) $v=v(r)$ where $r=|x|$.

(2) For each $T_m=\{x\,;\,2^{-m}\rho<|x|<2^{1-m}\rho\}$ $(m=1, 2, \cdots)$, v is harmonic in T_m.

(3) $v(2^{-m}\rho)=\int_{\partial B(0,2^{-m}\rho)} udS/\mathrm{vol}\,(\partial B(0, 2^{-m}\rho))$.

Using Proposition 2.1 we obtain

$$\sup_{B(0,\rho)}|u-v|\leqq C_3\Big(\rho^{2-n}\int_{B(0,2\rho)}|Du|^2\,dx\Big)^{1/2}\leqq C_4\varepsilon^{1/n}.$$

We estimate the energy of $u-v$ on $B(0, \rho)$ from above and below. By Lemma 3.2, we obtain

$$\int_{B(0,\rho)}|D(u-v)|^2\,dx\geqq\frac{1}{2}\int_{B(0,\rho)}|Du|^2\,dx. \tag{4.1}$$

Next we use the Gauss-Green theorem in each T_m to obtain

$$\int_{B(0,\rho)}|D(u-v)|^2\,dx=\sum_{m=1}^{\infty}\Big[\int_{S(r)}(u-v)(D_r u-v'(r))dS\Big|_{r=2^{-m}\rho}^{r=2^{1-m}\rho}-\int_{T_m}(u-v)\Delta(u-v)dx\Big],$$

where $S(r)=\partial B(0, r)$. The integral of the boundary term containing $v'(r)$ disappears because of (3). By Proposition 2.1, we have

$$\int_{B(0,\rho)}|D(u-v)|^2\,dx=\int_{S(\rho)}(u-v)D_r udS-\int_{B(0,\rho)}(u-v)\Delta udx.$$

Since u is a harmonic map, we have

$$-\int_{B(0,\rho)}(u-v)\Delta udx\leqq C_4\|A\|_\infty\varepsilon^{1/n}\int_{B(0,\rho)}|Du|^2\,dx, \tag{4.2}$$

where $\|A\|_\infty$ is the bound of the second fundamental form A. Thus, we obtain

$$\Big(\frac{1}{2}-C_4\|A\|_\infty\varepsilon^{1/n}\Big)\int_{B(0,\rho)}|Du|^2\,dx\leqq\int_{S(\rho)}(u-v)D_r udS.$$

We choose ε satisfying $C_4\|A\|_\infty\varepsilon^{1/n}\leqq 1/4$. Then, we have

$$\int_{B(0,\rho)}|Du|^2\,dx\leqq 4\Big(\int_{S(\rho)}|u-v|^2\,dS\Big)^{1/2}\Big(\int_{S(\rho)}|D_r u|^2\,dS\Big)^{1/2}.$$

We set $F(\rho)=\rho^{2-n}\int_{B(0,\rho)}|Du|^2\,dx$. From (3.1), we finally obtain

$$\rho^{-1}F(\rho)^2\leqq C_5F(2\rho)F'(\rho). \tag{4.3}$$

For $\rho\in(0, R/8)$ we integrate (4.3) from ρ to 2ρ. Using the fact that $F(\rho)$ is non-decreasing we have

$$F(\rho)\leqq\mu F(4\rho)\qquad\text{where } \mu=(C_5/(C_5+\log 2))^{1/2}<1.$$

We apply Lemma 8.23 in [2] to obtain

$$F(\rho)\leqq C_6\rho^\gamma\qquad\text{for } \rho\leqq R/8,\ \text{some } \gamma>0. \tag{4.4}$$

Combining (4.4) with Proposition 2.1, we have

$$|Du(x)| \leqq C_7 |x|^{-1+\gamma/2} \qquad \text{for } 0<|x| \leqq R/8.$$

This implies that u belongs to $W^{1,p}(B(0,R),\boldsymbol{R}^n)$ for some $p>n$. By the Sobolev imbedding theorem we have derived Hölder continuity of u.

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