26. *Semigroups of Differentiable Operators*

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(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1984)

§ 1. **Introduction.** Let A be the infinitesimal generator of a (C_0)-semigroup $\{T(t);\ t\geqq 0\}$ on a Banach space X, and let F be a Lipschitzian on X. It is known that $A+F$ generates a nonlinear semigroup $\{S(t);t\geqq 0\}$ on X, i.e.,

$$S(t)x=\lim_{\lambda\downarrow 0}(I-\lambda A-\lambda F)^{-[t/\lambda]}x \qquad (x\in X,\ t\geqq 0),$$

and that for every $x\in X$, $S(t)x$ is a (unique) mild solution to the semilinear equation

(1.1) $$(d/dt)u(t)=[A+F]u(t) \quad (t\geqq 0), \quad u(0)=x,$$

i.e., $u(t)=S(t)x$ is continuous in $t\geqq 0$ and satisfies the integral equation

(1.2) $$u(t)=T(t)x+\int_0^t T(t-s)Fu(s)\,ds \qquad (t\geqq 0).$$

(See [2].)

By $dF(x)$ we denote the derivative of F at x, i.e.,

$$dF(x)w=\lim_{h\downarrow 0}[F(x+hw)-Fx]/h \qquad (x,w\in X).$$

We say that F is continuously Gâteaux (resp. Fréchet) differentiable on X if $x\mapsto dF(x)$ is continuous on X in the strong (resp. uniform) operator topology.

The purpose of this paper is to investigate the relations between the continuous Gâteaux (or Fréchet) differentiability of F and $S(t)$. Our results (Theorems 1 and 2 in § 2) are closely related to [3] and [5], which discuss the Fréchet differentiability of $S(t)$ and the regularity of solutions to (1.1) and (1.2) in case that F is continuously Fréchet differentiable on X.

§ 2. **Theorems.** Our results are as follows:

Theorem 1. *Let F be continuously Gâteaux* (*resp. Fréchet*) *differentiable on X. Then we have*

(a) *$S(t)$ is continuously Gâteaux* (*resp. Fréchet*) *differentiable on X for each $t\geqq 0$, and $dS(t)(x)w$ is continuous in $(t,x,w)\in[0,\infty)\times X\times X$.*

(b) *The derivative of $S(t)$ is represented by*

(2.1) $$\begin{aligned} dS(t)(x)w&=\lim_{\lambda\downarrow 0}\prod_{i=1}^{[t/\lambda]}[I-\lambda A-\lambda dF(S(i\lambda)x)]^{-1}w\\ &=\lim_{\lambda\downarrow 0}\prod_{i=1}^{[t/\lambda]}[I-\lambda A-\lambda dF(J_\lambda^i x)]^{-1}w\\ &=\lim_{\lambda\downarrow 0}dJ_\lambda^{[t/\lambda]}(x)w \qquad (t\geqq 0;\ x,w\in X),\end{aligned}$$

where $J_\lambda=(I-\lambda A-\lambda F)^{-1}$, and the convergence is uniform on bounded interval.

(c) *If* $x \in D(A)$, *then* $S(t)x \in D(A)$ *for any* $t \geqq 0$, *and* $S(t)x$ *is a* C^1-*solution to* (1.1) *and satisfies*

$$(2.2) \qquad (d/dt)S(t)x = dS(t)(x)[A+F]x \quad (=[A+F]S(t)x) \quad (t \geqq 0).$$

(d) $dS(t)(x)w$ $(x, w \in X)$ *is a mild solution to*

$$(2.3) \qquad (d/dt)v(t) = [A + dF(S(t)x)]v(t) \quad (t \geqq 0), \quad v(0) = w,$$

i.e., $v(t) = dS(t)(x)w$ *satisfies the integral equation*

$$(2.4) \qquad v(t) = T(t)w + \int_0^t T(t-s)dF(S(s)x)v(s)\,ds \qquad (t \geqq 0).$$

(e) $U(t, s) = dS(t-s)(S(s)x)$ $(0 \leqq s \leqq t)$ *is a linear evolution operator on* X.

(f) $A + dF(S(t)x)$ *is the infinitesimal generator of* $U(t, s)$ *at* t, i.e.,

$$[A + dF(S(t)x)]w = \lim_{h\downarrow 0}[U(t+h, t)w - w]/h \qquad (w \in D(A)).$$

(g) *The derivative of* F *is represented by*

$$(2.5) \qquad \begin{aligned} dF(x)w &= \lim_{h\downarrow 0}[dS(h)(x)w - T(h)w]/h \\ &(= (d/dt)^+[dS(t)(x)w - T(t)w]_{t=0}), \end{aligned}$$

where the convergence is uniform in x *on compact* (*resp. bounded*) *set of* X *for each* $w \in X$.

Remark. It is known ([4]) that (c) does not hold in general.

Theorem 2. *F is continuously Gâteaux differentiable on X if and only if* (a) *in Theorem* 1 *and the following* (g′) *holds*:

(g′) *The limit of* (2.5) *exists for each* $x \in X$ *and* $w \in D(A)$, *and the convergence is uniform in* x *on compact set of* X.

§3. Proofs of theorems. We shall only give outlines of proofs. For details, we shall publish elsewhere with other results.

Proof of Theorem 1. (a) and (d) Let $V(t, x)$ be the bounded linear operator on X defined by $V(t, x)w = v(t)$, where $v(t)$ is a mild solution to (2.3). From (1.2) and (2.3), we have

$$\begin{aligned} &\|[S(t)(x+hw) - S(t)x]/h - V(t, x)w\| \\ &\leqq M\int_0^t \|[S(s)(x+hw) - S(s)x]/h - V(s, x)w\|\,ds \\ &\qquad + N\int_0^t \|F[S(s)x + hV(s, x)w] - FS(s)x - hdF(S(s)x)V(s, x)w\|/h\,ds, \end{aligned}$$

where M and N are positive constants independent of t in bounded interval. Using Gronwall's type estimate, we have the uniform convergence of $[S(t)(x+hw) - S(t)x]/h$ to $V(t, x)w$ on compact (resp. bounded) set of $[0, \infty) \times X \times X$ as $h \downarrow 0$.

(b) From the continuity of $dF(\cdot)$, $[A + dF(S(t)x)]w$ is continuous in t for each $w \in D(A)$, and for any $T \geqq 0$ there exists $\omega > 0$ such that $A + dF(S(t)x) - \omega I$ $(t \in [0, T])$ is linear m-dissipative. From this, we have the uniform convergence of the first equality in (2.1). (See [1].) It is easy to check that the difference between

$\prod_{i=1}^{[t/\lambda]}[I - \lambda A - \lambda dF(S(i\lambda)x)]^{-1}w$ and $\prod_{i=1}^{[t/\lambda]}[I - \lambda A - \lambda dF(J_\lambda^i x)]^{-1}w$

converges to 0 uniformly on each bounded interval as $\lambda \downarrow 0$, and that

$dJ_\lambda(x)$ is equal to $[I-\lambda A-\lambda dF(J_\lambda x)]^{-1}$. Consequently, (b) holds.

(c) It is shown ([4]) that $S'(0)x$ exists for each $x\in D(A)$ and is equal to $(A+F)x$, and that $S'(t)x$ $(x\in D(A))$ is equal to $(A+F)S(t)x$ if it exists. Let $h>0$ and $x\in D(A)$. We have

$$[S(t+h)x-S(t)x]/h=dS(t)(x)[S(h)x-x]/h$$
$$+\int_0^1 \{dS(t)[\theta S(h)x+(1-\theta)x]-dS(t)(x)\}[S(h)x-x]/h\,d\theta.$$

Letting $h\downarrow 0$, we have $S'^{+}(t)=dS(t)(x)[A+F]x$ for $x\in D(A)$. Since $dS(t)(x)[A+F]x$ is continuous in t, $S(t)x$ $(x\in D(A))$ is a C^1-solution to (1.1).

(e) follows from the chain rule of derivation.

(f) and (g) $U(t+h,t)w$ $(w\in D(A))$ satisfies

$$U(t+h,t)w=T(h)w+\int_0^h T(h-s)dF(S(t+s)x)U(t+s,t)w\,ds.$$

The second term of the right-hand side is differentiable at $h=0$ for any $w\in X$ and its derivative is $dF(S(t)x)w$. Letting $t=0$, we have (2.5). It is easy to check the uniform convergence. If $w\in D(A)$, then $T(h)w$ is differentiable in h. Thus we have (f).

To prove Theorem 2, we shall mention the following lemma without proof.

Lemma. *Let $f_\alpha(t)$ $(\alpha\in I)$ be a continuous X-valued function on $[0,T]$ with $f_\alpha(0)=0$ and differentiable at $t=0$. Assume that the convergence*

$$f'_\alpha(0)=\lim_{h\downarrow 0}[f_\alpha(h)-f_\alpha(0)]/h$$

is uniform in $\alpha\in I$ and $\{\|f'_\alpha(0)\|\}$ is bounded. Then we have

$$f'_\alpha(0)=\lim_{\lambda\downarrow 0}\lambda^{-2}\int_0^\tau \exp(-s/\lambda)f_\alpha(s)\,ds,$$

where the convergence is uniform in $\alpha\in I$ for some τ in $(0,T]$.

Proof of Theorem 2. For $\lambda>0$, we set

$$F_\lambda x=\lambda^{-2}\int_0^\tau \exp(-s/\lambda)[S(s)x-T(s)x]\,ds \qquad (x\in X).$$

Since $S(t)x-T(t)x$ is differentiable at $t=0$ for any $x\in X$, we have $Fx=\lim_{\lambda\downarrow 0}F_\lambda x$. It is clear that F_λ is continuously Gâteaux differentiable on X and its derivative is

$$dF_\lambda(x)w=\lambda^{-2}\int_0^\tau \exp(-s/\lambda)[dS(s)(x)w-T(s)w]\,ds.$$

Applying the lemma, we have that for each $w\in D(A)$

$G(x)w=\lim_{\lambda\downarrow 0}dF_\lambda(x)w$ uniformly in x on every compact set.

Since F_λ is Lipschitz continuous on X and $D(A)$ is dense in X, $G(x)$ is extended to a bounded linear operator $\overline{G(x)}$ defined on X. It follows that $x\mapsto\overline{G(x)}$ is continuous on X in the strong operator topology and $\overline{G(x)}$ is the Gâteaux derivative of F at x.

References

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