106. *Invariance of the Plurigenera of Algebraic Varieties under Minimal Model Conjectures*

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The purpose of this paper is to outline our recent results (see Theorems 1, 2 in § 3) on the behavior of plurigenera under projective deformation, provided that the minimal model conjectures (see § 1) are true. Here all the varieties are defined over the field of complex numbers. Details will be published elsewhere.

§ 1. Let X be a complete algebraic variety. A divisor D on X is called *nef*, if $D\cdot C\geqq 0$ for any curve C on X. The *numerical Kodaira dimension* for a nef Cartier divisor D is defined by

$$\nu(D):=\kappa_{num}(D):=\max\{d\,|\,D^d\not\approx 0\}.$$

If $\kappa(D)=\nu(D)$, D is called *good*. If $\kappa(D)=\dim X$ or $\nu(D)=\dim X$, then $\kappa(D)=\nu(D)=\dim X$. Such a D is called *big*.

For a normal variety X, K_X denotes the *canonical divisor class* of X and ω_X denotes the *dualizing sheaf* of X. For a Weil divisor D, if mD is Cartier for some integer m, then D is called ***Q**-Cartier*. If K_X is $\boldsymbol{Q}$-Cartier, then X is called a ***Q**-Gorenstein* variety. If any Weil divisors are $\boldsymbol{Q}$-Cartier, X is called ***Q**-factorial*. For a $\boldsymbol{Q}$-Gorenstein variety X, the smallest positive integer r such that rK_X is Cartier is called the index of X, denoted by index(X). Let X be a normal $\boldsymbol{Q}$-Gorenstein variety. For some (any) resolution $d: Y\to X$, if $K_Y=d^*K_X+\sum a_iE_i$, where $\sum E_i$ is a normal crossing d-exceptional divisor, then the singularity of X is called *terminal, canonical* or *log-terminal* according as $a_i>0$, $a_i\geqq 0$ or $a_i>-1$, for all i, (see [2] and [5]).

Let X be a normal projective variety with only canonical singularities. For an extremal ray R on X, there exists a morphism $\mathrm{cont}_R: X\to V$ called a contraction of R. For definitions and details, refer to Kawamata [2]. The type of $\mathrm{cont}_R: X\to V$ is one of the following cases:

(i) $\dim X>\dim V$ and cont_R has connected fibers.

(ii) cont_R is a birational morphism not isomorphic in codimension 1.

(iii) cont_R is a birational morphism isomorphic in codimension 1.

In the case (i), a general fiber F of the cont_R is a $\boldsymbol{Q}$-Fano variety, i.e., $-K_F$ is an ample $\boldsymbol{Q}$-Cartier divisor. In particular, $\kappa(X)=-\infty$. In the case (ii), if X has only $\boldsymbol{Q}$-factorial terminal singularities, then the exceptional locus of cont_R is a prime divisor and V has also only $\boldsymbol{Q}$-factorial terminal singularities. In this case, the contraction is called a *good contraction*. In the case (iii), V has only rational singularities. However, it is not $\boldsymbol{Q}$-Gorenstein any more. This contraction is called a *bad contraction*.

If X is a normal projective variety with only canonical singularities whose canonical divisor K_X is nef, then X is called a *minimal model*. There are some conjectures.

Minimal Model Conjecture. *For a given nonsingular projective variety X_0, there exists a minimal model $X_{\min}$ birationally equivalent to X_0, or a $\boldsymbol{Q}$-factorial projective variety X with only terminal singularities such that X is birationally equivalent to X_0 and X has an extremal ray whose type is* (i).

Conjecture M_1: *Let X be a projective variety with only $\boldsymbol{Q}$-factorial terminal singularities and let $f: X \to Z$ be a bad contraction. Then there exist a $\boldsymbol{Q}$-factorial projective variety X^+ with only terminal singularities and a birational morphism $f^+: X^+ \to Z$ such that f^+ is isomorphic in codimension* 1 *and K_{X^+} is f^+-ample.*

The birational mapping $X \cdots \to X^+$ is called an *elementary transformation* or a *flip* in short.

Conjecture M_2: *After a finite number of steps of flips, we have an extremal ray of type* (i) *or* (ii).

If M_1 and M_2 are true, then for a given surjective morphism $f: X_0 \to Z$ from a nonsingular projective variety X_0 onto a projective variety Z, there exists a $\boldsymbol{Q}$-factorial projective variety X over Z with only terminal singularities such that either K_X is relatively nef, or X has a relative extremal ray of type (i).

Given a minimal model X, a theorem of Kawamata [3] asserts the following.

If $\kappa(K_X) = \nu(K_X)$, i.e., *K_X is nef and good, then K_X is semi-ample.* Thus he proposes the **Goodness Conjecture**:

If X is a minimal model, then K_X is good.

§2. Lemma 1. *Let X be a normal variety with only log-terminal singularities and X_0 an effective Cartier divisor on X. Letting $X_0 = \sum a_i D_i$ be the irreducible decomposition, we set $D = \sum_{(a_i=1)} D_i$, which is a reduced Weil divisor on X. The normalization of D is denoted by $\sigma: X_1 \to D \subset X_0$. Then, for each integer $m \geqq 1$, there exists a natural homomorphism*

$$\psi_m : \sigma_* \mathcal{O}_{X_1}(mK_{X_1}) \longrightarrow \mathcal{O}_X(mK_X + mK_0) \otimes \mathcal{O}_{X_0}$$

which is an isomorphism at the generic points of D. Especially, we have an injection

$$H^0(X_1, \mathcal{O}_{X_1}(mK_{X_1})) \longrightarrow H^0(X_0, \mathcal{O}_X(mK_X + mX_0) \otimes \mathcal{O}_{X_0}).$$

Lemma 2. *Let Z be a projective variety with only canonical singularities, C a nonsingular projective curve and let $g: Z \to C$ be a proper surjective morphism with connected fibers. If K_Z is g-nef and K_{Z_t} is semi-ample where Z_t is the general (or generic) fiber, then K_Z is g-semi-ample.*

Lemma 3. *Let X be a quasi-projective variety with only log-terminal singularities, S a quasi-projective variety and let $f: X \to S$ be a proper surjective morphism. If K_X is f-semi-ample, then for $i \geqq 0$ and $\nu \geqq 1$, $R^i f_* \mathcal{O}_X(\nu K_X)$ is torsion free.*

§3. From the above lemmas, we can prove the following

Theorem 1. *Let $f: X \to S$ be a smooth proper morphism between quasi-projective varieties. Then every m-genus*

$$P_m(X_t) = h^0(X_t, \mathcal{O}_{X_t}(mK_{X_t}))$$

is independent of t, where $X_t = f^{-1}(t)$, if the Conjectures M_1, M_2 for $(\dim X_t + 1)$-dimensional varieties and the Goodness Conjecture for $(\dim X_t)$-dimensional varieties are true.

Theorem 2. *Let $f: X \to C$ be a proper surjective morphism from a nonsingular projective variety X onto a nonsingular projective curve C. Then for each $t \in C$ and a general fiber $X_\xi = f^{-1}(\xi)$, $\xi \in C$,*

$$\sum_{i=1}^{p} P_m(\Gamma_i) \leqq P_m(X_\xi) = \operatorname{rank} f_* \mathcal{O}_X(mK_X), \qquad \text{for } m \geqq 1,$$

where the Γ_i are irreducible components of $X_t = f^{-1}(t)$, if we assume that the Conjectures M_1, M_2 for $(\dim X)$-dimensional varieties and the Goodness Conjecture for $(\dim X_t)$-dimensional varieties are true.

By a result of Tsunoda [7] that solves the minimal model conjecture for fiber spaces of 3-folds, we obtain the following:

Theorem 3. *Let $f: X \to C$ be a proper surjective morphism with connected fibers from a 3-dimensional nonsingular projective variety X onto a nonsingular projective curve C. Then for each $t \in C$ and a general fiber $X_\xi = f^{-1}(\xi)$, $\xi \in C$,*

$$\sum_{i=1}^{p} P_m(\Gamma_i) \leqq P_m(X_\xi) = \operatorname{rank} f_* \mathcal{O}_X(mK_X), \qquad \text{for } m \geqq 1,$$

where the Γ_i are irreducible components of $X_t = f^{-1}(t)$.

References

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