105. *Family of Jacobian Manifolds and Characteristic Classes of Surface Bundles*

By Shigeyuki MORITA
Department of Mathematics, College of Arts and Sciences,
University of Tokyo

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1. Introduction. In our previous papers ([2], [3], [4]), we have defined characteristic classes for *surface bundles*, namely differentiable fibre bundles whose fibres are closed orientable surfaces, and investigated general properties of them. The purpose of the present note is to announce new results concerning them. More precisely we study cohomological properties of the natural map from a given surface bundle to its associated "family of Jacobian manifolds" and by using them, we derive a rather strong linear dependence relations among our characteristic classes.

2. Review of the definition of characteristic classes. We begin by recalling the definition of our characteristic classes of surface bundles very briefly (see [3] for details). Let $\pi: E \to X$ be an oriented surface bundle with fibre Σ_g=closed orientable surface of genus $g \geqq 2$ and let ξ be the tangent bundle of π, namely it is the subbundle of the tangent bundle of E consisting of vectors which are tangent to the fibres. We denote $e(\xi) \in H^2(E; \boldsymbol{Z})$ for the Euler class of ξ and define a cohomology class $e_i \in H^{2i}(X; \boldsymbol{Z})$ by $e_i = \pi_*(e^{i+1}(\xi))$ where $\pi_*: H^{2(i+1)}(E; \boldsymbol{Z}) \to H^{2i}(X; \boldsymbol{Z})$ is the Gysin homomorphism. Next let us choose a fibre metric on ξ so that each fibre $E_x = \pi^{-1}(x)$ $(x \in X)$ inherits a Riemannian metric. Now let η be the vector bundle over X whose fibre over $x \in X$ is $H^1(E_x; \boldsymbol{R})$. If we identify $H^1(E_x; \boldsymbol{R})$ with the space of harmonic 1-forms on E_x, then the $*$-operator on $H^1(E_x; \boldsymbol{R})$ satisfies $*^2 = -1$. Hence it induces a structure of complex g-dimensional vector bundle on η. Let $c_i \in H^{2i}(X; \boldsymbol{Z})$ be its Chern class.

The above definition can be made at the universal space level. Namely if we denote $\mathcal{M}_g$ and $\mathcal{M}_{g,*}$ respectively for the mapping class groups of Σ_g and Σ_g relative to the base point $* \in \Sigma_g$, then the natural exact sequence

$$1 \longrightarrow \pi_1(\Sigma_g) \longrightarrow \mathcal{M}_{g,*} \xrightarrow{\pi} \mathcal{M}_g \longrightarrow 1$$

serves as the universal Σ_g-bundle and we have the universal cohomology classes

$$e \in H^2(\mathcal{M}_{g,*}; \boldsymbol{Z})$$
$$e_i = \pi_*(e^{i+1}), \qquad c_i \in H^{2i}(\mathcal{M}_g; \boldsymbol{Z}).$$

We also write e_i for the class $\pi^*(e_i) \in H^{2i}(\mathcal{M}_{g,*}; \boldsymbol{Z})$ for simplicity. There are several relations among these classes as explained in [3]. The main purpose of this note is to give a new relation among them.

3. Family of Jacobian manifolds. Let $J = \bigcup_{x \in X} J_x$, where $J_x = H_1(E_x; \boldsymbol{R})/H_1(E_x; \boldsymbol{Z})$ is the "Jacobian manifold" of E_x. The natural projection $\pi: J \to X$ admits a canonical structure of a *flat* T^{2g}-bundle whose structure group is $Sp(2g; \boldsymbol{Z})$. Hence there is a closed 2-form ω on J whose restriction to the fibre T^{2g} is the symplectic form. It turns out that the cohomology class of 2ω lifts to a canonical integral class of J which we denote by $\Omega \in H^2(J; \boldsymbol{Z})$. Now assume that our surface bundle $\pi: E \to X$ admits a cross section $s: X \to E$. Then we can define a fibre preserving natural map $j: E \to J$ whose restriction to each fibre E_x, $j_x: E_x \to J_x$, is the period map of E_x relative to the base point $s(x) \in E_x$. Let $e \in H^2(E; \boldsymbol{Z})$ be the Euler class of the surface bundle $\pi: E \to X$ as defined in §2 and let $\nu \in H^2(E; \boldsymbol{Z})$ be the cohomology class defined by the submanifold $s(X)$ of E of codimension 2. Then we have

Theorem 1. $j^*(\Omega) = 2\nu - e - \pi^* s^*(e)$.

Remark. The above theorem holds for all genus g.

4. Reformulation of Theorem 1 in terms of group cohomology. Theorem 1 for the case $g \geqq 2$ is proved at the universal space level roughly as follows. First of all we define a group $\overline{\mathcal{M}}_{g,*}$ by the following pull-back diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,*} & \xrightarrow{\bar{\pi}} & \mathcal{M}_{g,*} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \mathcal{M}_{g,*} & \longrightarrow & \mathcal{M}_g . \end{array}$$

Equivalently $\overline{\mathcal{M}}_{g,*}$ is the semi-direct product $\pi_1(\Sigma_g) \rtimes \mathcal{M}_{g,*}$ where $\mathcal{M}_{g,*}$ acts on $\pi_1(\Sigma_g)$ naturally. The exact sequence

$$1 \longrightarrow \pi_1(\Sigma_g) \longrightarrow \overline{\mathcal{M}}_{g,*} \underset{s}{\overset{\pi}{\rightleftarrows}} \mathcal{M}_{g,*} \longrightarrow 1$$

serves as the universal Σ_g-bundle with cross sections, where s is the "diagonal homomorphism". We have four cohomology classes

$$\pi^*(e), \quad \pi^*(e_1), \quad \bar{\pi}^*(e), \quad \nu \in H^2(\overline{\mathcal{M}}_{g,*}; \boldsymbol{Z}).$$

Next let $\overline{Sp}(2g; \boldsymbol{Z}) = H_1(\Sigma_g; \boldsymbol{Z}) \rtimes Sp(2g; \boldsymbol{Z})$, where $Sp(2g; \boldsymbol{Z})$ acts on $H_1(\Sigma_g; \boldsymbol{Z})$ naturally. There is a natural homomorphism

$$\rho: \overline{\mathcal{M}}_{g,*} \longrightarrow \overline{Sp}(2g; \boldsymbol{Z})$$

which should be considered as the universal model for the map $j: E \to J$. Now the cohomology class $\Omega \in \overline{Sp}(2g; \boldsymbol{Z})$ mentioned before is the one represented by the cocycle (we use the same letter)

$$\Omega((x, A), (y, B)) = x \cdot Ay$$

where (x, A), $(y, B) \in H_1(\Sigma_g; \boldsymbol{Z}) \times Sp(2g; \boldsymbol{Z})$ and $x \cdot Ay$ denotes the intersection number of x and Ay. With these notations we have

Theorem 2. $\rho^*(\Omega)=2\nu-\pi^*(e)-\bar{\pi}^*(e)\in H^2(\overline{\mathcal{M}}_{g,*};\boldsymbol{Z})$.

Theorem 1 follows from Theorem 2 as a direct consequence. The main point in the proof of Theorem 2 consists of identifying two naturally defined cohomology classes in $H^1(\mathcal{M}_{g,1};H^1(\Sigma_g;\boldsymbol{Z}))$, where $\mathcal{M}_{g,1}$ denotes the mapping class group of $\Sigma_g-\mathring{D}^2$.

5. New relations among characteristic classes of surface bundles. First of all we have

$$\Omega^{g+1}=0 \qquad \text{in } H^{2(g+1)}(\overline{Sp}(2g;\boldsymbol{Z});\boldsymbol{Q}).$$

The best way to see this is to recall that geometrically Ω is represented by twice of the form ω, because it is then clear that ω^{g+1} vanishes identically. We can also estimate the order of Ω^{g+1} in $H^{2(g+1)}(\overline{Sp}(2g;\boldsymbol{Z});\boldsymbol{Z})$. Theorem 2 combined with the above fact implies that

$$(2\nu-\pi^*(e)-\bar{\pi}^*(e))^{g+1}=0 \qquad \text{in } H^{2(g+1)}(\overline{\mathcal{M}}_{g,*};\boldsymbol{Q}).$$

If we apply the Gysin homomorphisms

$$\pi_*: H^*(\overline{\mathcal{M}}_{g,*};\boldsymbol{Q})\longrightarrow H^{*-2}(\mathcal{M}_{g,*};\boldsymbol{Q})$$

and

$$\pi_*: H^*(\mathcal{M}_{g,*};\boldsymbol{Q})\longrightarrow H^{*-2}(\mathcal{M}_g;\boldsymbol{Q})$$

successively to the above equation, using the relations

$$\nu^2=\nu\pi^*(e)=\nu\bar{\pi}^*(e), \qquad \pi_*\bar{\pi}^*(e^{i+1})=e_i,$$

we obtain

Corollary 3.

(i) $$e_{g+h-1}+\binom{g+h}{1}e_{g+h-2}e+\cdots+\binom{g+h}{g+h-2}e_1e^{g+h-2}+\left\{\binom{g+h}{g+h-1}(2-2g)-2^{g+h}\right\}e^{g+h-1}=0$$
$$\text{in } H^{2(g+h-1)}(\mathcal{M}_{g,*};\boldsymbol{Q}) \quad (h\geqq 1).$$

(ii) $$e_{g+h-1}e_{k-1}+\binom{g+h}{1}e_{g+h-2}e_k+\cdots+\binom{g+h}{g+h-2}e_1e_{g+h+k-3}+\left\{\binom{g+h}{g+h-1}(2-2g)-2^{g+h}\right\}e_{g+h+k-2}=0$$
$$\text{in } H^{2(g+h+k-2)}(\mathcal{M}_g;\boldsymbol{Q}) \quad (h\geqq 1,\ k\geqq 0).$$

These relations, when combined with the previously known ones, are very strong. For example if $g=3$, only the class $e_1\in H^2(\mathcal{M}_3;\boldsymbol{Q})$ remains to be non trivial. It would be very interesting if one can determine whether $H^*(\mathcal{M}_3;\boldsymbol{Q})\cong H^*(S^2;\boldsymbol{Q})$ or not.

6. Concluding remarks. (i) If the surface bundle $\pi: E\to X$ admits a holomorphic structure, namely if both of E and X are (possibly non compact) complex manifolds and π, s are holomorphic, then all the terms which appear in Theorem 1 can be considered as holomorphic line bundles over E. For example, for the class Ω we have the line bundle over J whose restriction to each fibre J_x is the one defined by twice of the zero locus of the Riemann theta function. It seems to be reasonable to conjecture that Theorem 1 remains to be true in this refined form. One can even ask whether Corollary 3 remains to be true in the Chow ring $A^*(M_g)$ of Mumford [5] or not.

(ii) The cohomology class $\Omega \in H^2(\overline{Sp}(2g\,;Z)\,;Z)$ is not divisible by 2 as an integral class except for the case $g=1$.

(iii) It turns out that there is a close relationship between Johnson's conjecture on the homology of the Torelli group ([1]) and the non-triviality of some of the characteristic classes restricted to the Torelli group. In fact one of the motivations for the present work was to understand the connection of Johnson's work with ours.

(iv) We can also apply Sullivan's construction of non abelian Jacobians ([6]) fibre-wise to obtain flat fibre bundles over X with various nilmanifolds as fibres. One can expect that one could obtain more informations from them.

(v) The details of proofs of Theorem 2 and Corollary 3 together with the above points will appear elsewhere.

References

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