100. A Note on Sums and Maxima of Independent, Identically Distributed Random Variables

By Yuji KASAHARA
Institute of Mathematics, University of Tsukuba

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§1. **Introduction.** Let $X_1, X_2, \cdots$ be i.i.d. (independent, identically distributed) random variables and put $S_n = X_1 + \cdots + X_n$, and $M_n = \max(X_1, \cdots, X_n)$ $n=1, 2, \cdots$. Chow-Teugels [2] studied the joint limiting distributions of (S_n, M_n) as $n \to \infty$ after suitable normalizations. In this note we will consider this problem using the theory of point processes and generalize the result of [2] to a functional limit theorem for the sums and the maxima of triangular arrays of i.i.d. random variables.

§2. **Main theorem.** Let $\{\xi_{nk}\}_{k=1}^{\infty}$ be i.i.d. random variables with distribution function $F_n(x)$, $n=1, 2, \cdots$. Throughout this paper we assume that for suitably chosen constants A_n, $n=1, 2, \cdots$ and a nondegenerate distribution function $F(x)$ we have

(2.1) $\lim_{n\to\infty} P[\sum_{j=1}^{n} \xi_{nj} - A_n \leqq x] = F(x)$ at all continuity points of $F(x)$.

The characteristic function $\phi(\theta)$ of $dF(x)$ has the following representation.

$$\phi(\theta) = \exp\left[\gamma\theta - \frac{1}{2}\sigma^2\theta^2 + \int\{e^{i\theta x} - 1 - i\theta x I(|x| \leqq \delta)\}\mu(dx)\right] \tag{2.2}$$

where $\gamma \in \boldsymbol{R}$, $\sigma^2 \geqq 0$, $\int \min(1, x^2)\mu(dx) < \infty$ and $\delta > 0$ is chosen so that $\mu\{\pm\delta\} = 0$.

It is well known that (2.1) implies a functional limit theorem; the process

$$\xi_n(t) = \sum_{j \leqq nt} \xi_{nj} - A_n[nt]/n \tag{2.3}$$

converges in law to the Lévy process $\xi(t)$ with characteristic (2.2) over the Skorohod function space $D([0, \infty); \boldsymbol{R})$ endowed with the J_1-topology (see [5] for the definition). We also assume that there exist constants $B_n > 0$, C_n, $n=1, 2, \cdots$ and nondegenerate distribution function $G(x)$ such that

(2.4) $\lim_{n\to\infty} P[B_n \max_{k\leqq n} \xi_{nk} - C_n \leqq x] = G(x)$ at all continuity points of $G(x)$ (see Lemma 4.1.).

(As we will see later, if $\mu(0, \infty) > 0$ then this condition is automatic from (2.1) with $B_n = 1$, $C_n = 0$.) It is also well known that (2.4) implies a functional limit theorem: Define

$$m_n(t)=\begin{cases} B_n \max_{k\leq nt} \xi_{nk}-C_n, & t\geqq 1/n \\ m_n(1/n), & 0<t<1/n. \end{cases} \tag{2.5}$$

Then we have that $\{m_n(t)\}_{t>0}$ converges in law over $D((0,\infty);\boldsymbol{R})$ to a nondecreasing process $m(t)$ with marginals as follows.

$$\begin{aligned} P[m(t_j)\leqq x_j,\ j=1,\cdots,n] &\\ =G(x_1)^{t_1}G(x_2)^{t_2-t_1}\cdots G(x_n)^{t_n-t_{n-1}}&, \\ \text{for } 0<t_1<\cdots<t_n,\ x_1<x_2<\cdots<x_n,\ n=1,2,\cdots&. \end{aligned} \tag{2.6}$$

$\{m(t)\}$ is called the extremal process associated with $G(x)$. We now consider the joint limiting process of $\{(\xi_n(t), m_n(t))\}$. Since $m_n(t) =\sup_{s\leqq t}\{\xi_n(s)-\xi_n(s-)\}$ $(t>1/n)$, applying the continuity theorem (see e.g. [1] p. 31) we see that $\{(\xi_n(t), m_n(t))\}$ converges in law to $\{(\xi(t), \max_{s\leqq t}(\xi(s)-\xi(s-)))\}$. If $\mu(0,\infty)>0$ then $\max_{s\leqq t}\{\xi(s)-\xi(s-)\}$ is not trivial and therefore we have a complete answer to the joint convergence of $(\xi_n(t), m_n(t))$ with $B_n=1$, $C_n=0$. Thus we need to consider only the case where

$$\mu(0,\infty)=0. \tag{2.7}$$

Our main theorem is

Theorem 1. *Assume* (2.1), (2.4) *and* (2.7). *Let* ξ_n, m_n, ξ *and* m *be as before. Then* $\{(\xi_n(t), m_n(t))\}_{t>0}$ *converges in law to* $\{(\tilde{\xi}(t), \tilde{m}(t))\}$ *as* $n\to\infty$ *in* $D((0,\infty);\boldsymbol{R}^2)$ *endowed with the* J_1*-topology, where* $\{\tilde{\xi}(t)\}$ *and* $\{\tilde{m}(t)\}$ *are independent and are identical in law to* $\{\xi(t)\}$ *and* $\{m(t)\}$, *respectively.*

§ 3. Outline of the proof.

Let p be a Poisson point process on $\{(t,x)\,|\,t>0,\ x\in\boldsymbol{R}\backslash\{0\}\}$ with intensity measure $\hat{N}_p(dt\,dx)=dt(dx/x^2)$, and let $N_p(ds\,dx)$ denote the counting measure of p. We also put $\tilde{N}_p(ds\,dx)=N_p(ds\,dx)-\hat{N}_p(ds\,dx)$. We refer to the textbook of Ikeda and Watanabe [4] for the details of definitions, notations and fundamental results of (Poisson) point processes.

Proposition. *Assume* (2.1) *and* (2.4) *but we drop the condition* (2.7). *Put*

$$f(x)=\begin{cases} \inf\{t>0\,|\,\mu[t,\infty)<1/x\}, & x>0 \\ \sup\{t<0\,|\,\mu(-\infty,t)<-1/x\}, & x<0 \end{cases} \tag{3.1}$$

and

$$g(x)=G^{-1}(e^{-1/x}),\qquad x>0. \tag{3.2}$$

Then, $\{(\xi_n(t), m_n(t))\}_{t>0}$ *converges in law to*

$$(\gamma t+\sigma B(t)+\zeta_\delta(t)+\eta_\delta(t),\ g(\max_{s\leqq t} p(s))) \tag{3.3}$$

where γ, σ *and* δ *are the constants in* (2.2), $B(t)$ *is a standard Brownian motion independent of the Poisson point process* p *and where*

$$\zeta_\delta(t)=\int_0^{t+}\int_{\{|f(x)|\leqq\delta\}} f(x)\tilde{N}_p(ds\,dx), \tag{3.4}$$

(3.5) $$\eta_\delta(t)=\int_0^{t+}\int_{\{|f(x)|>\delta\}} f(x)N_p(ds\,dx), \qquad t>0.$$

This proposition can be proved by using the idea of [3] and we omit the details. Theorem 1 is an easy consequence of this proposition; if $\mu(0,\infty)=0$ then we have $f(x)=0$ on $(0,\infty)$. Therefore, the (stochastic) integrals in (3.4) and (3.5) are in fact functionals of the restriction p^- of p to the lower half plane $\{x<0\}$ while $\max_{s\leqq t} p(s)$ depends only on the restriction p^+ of p to the upper half plane. Since p^+ and p^- are independent, we see that the first and the second components of (3.3) are mutually independent, which prove the theorem.

§ 4. **Supplement.** By a slight modification of the proof, Theorem 1 may be extended to all order statistics. Let $M^k(n)$ denote the k-th largest among $\{\xi_{n1},\cdots,\xi_{nn}\}$. For $k>n$, we define $M^k(n)=M^k(k)$ for convenience. (Thus $M^k(n)$ is nondecreasing in n.)

Theorem 2. *Assume* (2.1), (2.4) *and* (2.7) *and let* $g(x)=G^{-1}(e^{-1/x})$, $x>0$ *as before. Define* $m_n^{(k)}(t)=B_nM^k([nt])-C_n$, $t>0$, n, $k\geqq 1$. *Then, for* $N\geqq 1$, $\{(\xi_n(t), m_n^{(1)}(t),\cdots,m_n^{(N)}(t))\}_{t>0}$ *converges in law in* $D((0,\infty);\boldsymbol{R}^{1+N})$ *to* $\{(\tilde{\xi}(t), g(J^{(1)}(t)),\cdots,g(J^{(N)}(t))\}_{t>0}$ *as* $n\to\infty$, *where* $J^{(k)}(t)$ *denotes the* k*-th largest among* $\{p(s);s\leqq t, s\in D_p\}$, *and where* $\{\tilde{\xi}(t)\}$ *is a process which is independent of* p *and is identical in law to* $\{\xi(t)\}$. *(*p *is the Poisson point process in* § 3.)

We can also consider similar problems for the case where the sums and the maxima are based on different arrays of i.i.d. random variables. Let $\{(\xi_{nk},\zeta_{nk})\}_{k=1}^\infty$ be i.i.d. random vectors such that $\{\xi_{nk}\}$ satisfies (2.1) and that (2.4) holds replacing $\{\xi_{nk}\}$ by $\{\zeta_{nk}\}$. Let $\xi_n(t)$ be as in (2.3) and define $m_n(t)=B_n\max_{k\leqq nt}\zeta_{nk}-C_n$. If $\mu(dx)$ vanishes identically, we have a result similar to Theorem 1;

Theorem 3. *Assume that* $\mu(\boldsymbol{R}\setminus\{0\})=0$ *as well as above assumption. Then the assertion of Theorem* 1 *is still valid.*

Finally, we give a necessary and sufficient condition for (2.4).

Lemma 4.1. (2.4) *is equivalent to the following condition.*

(4.1) $$\lim_{n\to\infty} B_nF_n^{-1}\left(1-\frac{1}{nx}\right)-C_n=(gx) \quad \textit{at all continuity points } x>0,$$

where $g(x)=G^{-1}(e^{-1/x})$ *as before.*

Proof. It is easy to see that (2.4) is equivalent to

(4.2) $$\lim_{n\to\infty} n\{F_n((C_n+x)/B_n)-1\}=\log G(x).$$

By considering the inverse functions of the both sides of (4.2) we have the assertion after a change of the variable $(x\to -1/x)$.

References

[1] Billingsley, P.: Convergence of probability measures. John Wiley & sons, Inc., New York (1968).

[2] Chow, T. L., and Teugels, J. L.: The sum and the maximum of i.i.d. random variables. Proc. Second Prague Symp. on Asymptotic Stat., North-Holland, Amsterdam (1979).

[3] Durret, R., and Resnick, S. I.: Functional limit theorem for dependent variables. Ann. Probab., **6**, 829–846 (1978).

[4] Ikeda, N., and Watanabe, S.: Stochastic Differential Equations and Diffusion Processes. North-Holland, Amsterdam/Kodansha, Tokyo (1981).

[5] Lindvall, T.: Weak convergence of probability measures and random functions in the function space $D[0, \infty)$. J. Appl. Probab., **10**, 109–121 (1973).