94. On Zero-divisors in Reduced Group Rings over Ordered Groups

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In this note, a ring will mean (not necessarily commutative) ring with unity 1. A ring R is said to be *reduced*, if R has no nonzero nilpotent element. A group G $(\neq 1)$ is called *ordered*, if it admits strict linear ordering $<$ such that $g<h$ implies $gk<hk$, $kg<kh$ for all $k \in G$ (cf. Passman [1]). Our aim is to prove the following theorem.

Theorem. *Let R be a reduced ring and G an ordered group. Let α, β be elements of the group ring RG: $\alpha=\sum_{i=1}^{n} a_i g_i$, $\beta=\sum_{j=1}^{m} b_j h_j$, where $a_i, b_j \in R$ $(a_i \neq 0, b_j \neq 0)$, and $g_1, \cdots, g_n$ and $h_1, \cdots, h_m$ are respectively mutually distinct elements of G. Then we have $\alpha\beta=0$ if and only if $a_i b_j=0$ for all $i=1, \cdots, n$, $j=1, \cdots, m$.*

For the proof we use the following simple lemma on a reduced ring R.

Lemma. *If a, b are elements of a reduced ring R, $aba=0$ implies $ba=0$. (In particular, $ab=0$ implies $ba=0$.)*

Proof. If $aba=0$, we have $(ba)^2=baba=0$. As R has no nonzero nilpotent element, this implies $ba=0$.

Proof of the Theorem. The if-part being obvious, we have only to prove the only-if-part: $\alpha\beta=0$ implies $a_i b_j=0$. We have nothing to prove, if $n=m=1$. So suppose $n \geq 2$, $m \geq 2$. As G is ordered and $g_1, \cdots, g_n$ and $h_1, \cdots, h_m$ are respectively mutually distinct, we may assume $g_1<\cdots<g_n$, $h_1<\cdots<h_m$. We have

$$\alpha\beta=\sum_{1 \leq i \leq n, 1 \leq j \leq m} a_i b_j g_i h_j=0 \tag{1}$$

and $g_1 h_1$ is the "smallest among $g_i h_j$" i.e. we have $g_1 h_1<g_i h_j$ for any i, j with $1<i$, $1<j$. Thus we should have $a_1 b_1=0$.

To simplify the further description of our proof, we shall use the following expressions on pairs of indices (i, j), (i', j'), $\cdots$ where $i, i', \cdots \in \{1, 2, \cdots, n\}$, $j, j', \cdots \in \{1, 2, \cdots, m\}$. These mn pairs are ordered according to the "magnitudes" of $g_i h_j$, $g_{i'} h_{j'}$, $\cdots$; we shall say namely (i, j) is *smaller* than (i', j') and write $(i, j)<(i', j')$ when $g_i h_j<g_{i'} h_{j'}$; (i, j) is called *equivalent* to (i', j'), written $(i, j) \sim (i', j')$, when $g_i h_j= g_{i'} h_{j'}$. From $i<i'$ follows obviously $(i, j)<(i', j)$, and from $(i, j)<(i', j')$, $(i', j') \sim (i'', j'')$ follows $(i, j)<(i'', j'')$. We shall prove $a_i b_j=0$ following the "magnitudes" of (i, j) beginning from the smallest pair $(1, 1)$. A pair (i, j) will be called *settled*, if $a_i b_j=0$ has been proved. Thus $(1, 1)$

is settled, and in proving $a_{i_0}b_{j_0}=0$ for a fixed pair (i_0, j_0), we can obviously assume that all (i, j) are settled for $(i, j)<(i_0, j_0)$. Let $\{(i_1, j_1), \cdots, (i_p, j_p)\}$ be the set of all unsettled pairs which are equivalent to (i_0, j_0). From (1) follows

$$(2) \qquad a_{i_1}b_{j_1}+\cdots+a_{i_p}b_{j_p}=0.$$

We have nothing more to prove if $p=1$. So let $p\geq 2$ and $i_1<i_2<\cdots<i_p$. Then we have for $k\geq 2$ $(i_1, j_k)<(i_k, j_k)\sim(i_0, j_0)$ so that (i_1, j_k) is settled by our assumption and $a_{i_1}b_{j_k}=0$ whence follows $b_{k_j}a_{i_1}=0$ by Lemma. Multiplying (2) by a_{i_1} from right, we obtain $a_{i_1}b_{j_1}=0$, i.e. (i_1, j_1) is settled and we can proceed further.

In the following Corollaries, the notations $R, G, \alpha=\sum a_i g_i$, $\beta=\sum b_j h_j$ will have the same meanings as in the Theorem.

Corollary 1. *If α is a zero-divisor of RG, then a_i are zero-divisors of R.*

Corollary 2. *$a_i=b_j$ $(\neq 0)$ can not take place for any $i=1, \cdots, n$, $j=1, \cdots, m$.*

Proof. If $a_i=b_j$, then $a_ib_j=a_i^2=0$ and $a_i=0$.

Corollary 3. *RG has no non-trivial idempotent.*

Proof. Let e be an idempotent of RG. Then $e^2=e$, $e(e-1)=0$, which implies $e=0$ or $e=1$ by virtue of Corollary 2.

Remark. If G is not torsion free, then RG contains elements α, β with $\alpha\beta=0$, such that coefficients of g_i, h_j in α, β are not zero-divisors; e.g. $\alpha=g-1$, $\beta=1+g+\cdots+g^{n-1}$ where $g^n=1$.

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Reference

[1] Passman, D. S.: Infinite Group Rings. Pure and Applied Mathematics; a series of monographs and text books. vol. 6, M. Dekker (1971).