85. *Extended Epstein's Zeta Functions over CM-fields*[*)]

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1. Introduction and statement of the results. The purpose of this note is to establish a relation between a series which derives from totally positive definite binary quadratic forms of discriminant Δ over a totally real algebraic number field F and Dedekind's Zeta function of CM-field $F(\sqrt{\Delta})$. In the case of Q, it has been done in [6, §4].

Let F be a totally real algebraic number field of degree n, $\mathfrak{o}_F$ the ring of integers in F, U_F the unit group of $\mathfrak{o}_F$ and $\Gamma=PSL_2(\mathfrak{o}_F)$. We assume the class number of F will be one in narrow sense. For any totally negative element Δ in $\mathfrak{o}_F$, denote by K the totally imaginary quadratic extention $F(\sqrt{\Delta})$ over F. Let Φ be the set of totally positive definite binary quadratic forms of discriminant Δ with $\mathfrak{o}_F$-coefficients. We consider Γ operates on Φ by

$$^{\sigma}\phi(x, y)=\phi(\alpha x+\gamma y,\ \beta x+\delta y), \qquad \left(\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right).$$

We define

$$(1)\qquad \zeta(s, \Delta)=\sum_{\phi\in\Phi/\Gamma}\ \sum_{(\mu,\nu)\in X/\mathrm{Aut}(\phi)} N_F(\phi(\nu, -\mu))^{-s} \qquad (\mathrm{Re}\,(s)>1).$$

Here, $X=\{\mathfrak{o}_F\times\mathfrak{o}_F-(0, 0)\}/U_F$, $\mathrm{Aut}\,(\phi)=\{\sigma\in\Gamma\,;\,{}^{\sigma}\phi=\phi\}$. Then $\zeta(s, \Delta)$ converges absolutely if $\mathrm{Re}\,(s)>1$, and uniformly if $\mathrm{Re}\,(s)\geqq 1+\varepsilon$ $(\varepsilon>0)$. So $\zeta(s, \Delta)$ is a holomorphic function in that region. It has been known from [3], [6] that $\zeta(s, \Delta)$ can be continued meromorphically to the whole plane and has a simple pole at $s=1$ because the first summation of (1) is a finite sum. We denote by D the discriminant of K over F, and by Δ_0 a totally negative integer such that $(\Delta_0)=D$. For a prime ideal $\mathfrak{p}$, put $\alpha_{\mathfrak{p}}=(1/2)(\mathrm{ord}_{\mathfrak{p}}(\Delta)-\mathrm{ord}_{\mathfrak{p}}D)$ and $\nu_{\mathfrak{p}}=\mathrm{ord}_{\mathfrak{p}}D$. For an even prime ideal $\mathfrak{p}$, let $e_{\mathfrak{p}}$ be the ramification index of $\mathfrak{p}$ in F. If $\mathfrak{p}$ ramifies in K, we define a non-negative integer $k_{\mathfrak{p}}$ by

$$\max\{0\leqq k_{\mathfrak{p}}\leqq(\nu_{\mathfrak{p}}/2)+1\,;\ x^2\equiv\Delta_0 \bmod \mathfrak{p}^{2e_{\mathfrak{p}}+2k_{\mathfrak{p}}} \text{ is solvable for } x\in\mathfrak{o}_F\},$$

otherwise, we put $k_{\mathfrak{p}}=0$. We say Δ is exceptional if $k_{\mathfrak{p}}\geqq 1$.

Theorem. *For a non-exceptional Δ, if $\alpha_{\mathfrak{p}}\geqq 0$ for all $\mathfrak{p}$, we have*

$$(2)\qquad \zeta(s, \Delta)=\zeta_K(s)\sum_{\mathfrak{n}\mid\mathfrak{f}}\mu(\mathfrak{n})\chi_{\Delta}(\mathfrak{n})N_F(\mathfrak{n})^{-s}\sigma_{1-2s}(\mathfrak{f}/\mathfrak{n}),$$

*) This work was started by the author while visiting at The University of Washington, Seattle, U.S.A.

otherwise $\zeta(s, \Delta)=0$.

Here $\zeta_K(s)$ *is Dedekind's zeta function of* K, $\mathfrak{f}=\prod_{\mathfrak{p}} \mathfrak{p}^{\alpha_{\mathfrak{p}}}$, $\mathfrak{n}$ *runs over all divisors of* $\mathfrak{f}$, $\mu(\mathfrak{n})$ *is Möbious' function over* $\mathfrak{o}_F$, $\sigma_s(\mathfrak{n})=\sum_{\mathfrak{m}|\mathfrak{n}} N_F(\mathfrak{m})^s$ *and* $\chi_\Delta(\mathfrak{n})$ *is the character attached to* K *over* F.

Using the functional equation of $\zeta_K(s)$, we have

Corollary. *For a non-exceptional* Δ *with all* $\alpha_{\mathfrak{p}}\geqq 0$, *we have a functional equation*

$$(3)\qquad \Lambda(s, \Delta)=\Lambda(1-s, \Delta),$$

where

$$(4)\qquad \Lambda(s, \Delta)=\gamma(s, \Delta)\zeta(s, \Delta),$$

$$(5)\qquad \gamma(s, \Delta)=(2\pi)^{-ns}\Gamma(s)^n(|N_F(\Delta)|D_F^2)^{s/2},$$

D_F *is the discriminant of* F.

Remark. For an exceptional Δ, the theorem should receive slight modifications. For an even prime ideal $\mathfrak{p}$, the case $k_{\mathfrak{p}}\geqq 1$ occurs only if $\nu_{\mathfrak{p}}$ is an even number, say $\nu_{\mathfrak{p}}=2m_{\mathfrak{p}}$. Put $\alpha'_{\mathfrak{p}}=\alpha_{\mathfrak{p}}+\min(k_{\mathfrak{p}}, m_{\mathfrak{p}})$ and $\chi_\Delta(\mathfrak{p})=0$, -1 or 1, according to $k_{\mathfrak{p}}<m_{\mathfrak{p}}$, $k_{\mathfrak{p}}=m_{\mathfrak{p}}$ or $k_{\mathfrak{p}}>m_{\mathfrak{p}}$. Besides, put $\mathfrak{f}'=\prod_{\mathfrak{p}=\text{even}} \mathfrak{p}^{\alpha'_{\mathfrak{p}}}\times\prod_{\mathfrak{p}=\text{odd}} \mathfrak{p}^{\alpha_{\mathfrak{p}}}$. In this case, $\zeta(s, \Delta)$ vanishes unless $\alpha'_{\mathfrak{p}}\geqq 0$ for all even prime ideals $\mathfrak{p}$ and $\alpha_{\mathfrak{p}}\geqq 0$ for all odd prime ideals $\mathfrak{p}$. Then we have

$$(6)\qquad \zeta(s, \Delta)=\zeta_K(s)\prod_{\mathfrak{p}}(1-\chi_\Delta(\mathfrak{p})N_F(\mathfrak{p})^{-s})^{-1} \times\sum_{\mathfrak{n}|\mathfrak{f}'}\mu(\mathfrak{n})\chi_\Delta(\mathfrak{n})N_F(\mathfrak{n})^{-s}\sigma_{1-2s}(\mathfrak{f}'/\mathfrak{n}),$$

where $\mathfrak{p}$ runs over all even prime ideals such that $k_{\mathfrak{p}}\geqq m_{\mathfrak{p}}$.

2. The sketch of proof. Let Δ be non-exceptional and $\alpha_{\mathfrak{p}}\geqq 0$ for all $\mathfrak{p}$. We transform $\zeta(s, \Delta)$ in (1) to

$$(6)\qquad \sum_{x\in X/\Gamma}\sum_{\phi\in\Phi/\Gamma_x} N_F(\phi(x))^{-s},$$

where Γ_x is the isotropy subgroup of x in Γ. Any Γ-orbit in Φ contains an element of type $(0, r)U_F$ $(r\in\mathfrak{o}_F-\{0\})$, therefore there is a one-to-one correspondence between Γ-inequivalence classes in X and integral ideals in $\mathfrak{o}_F$. For $x=(0, r)\in X$, the isotropy subgroup of x becomes

$$\Gamma_\infty=\left\{\begin{pmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{pmatrix};\ \alpha\in U_F,\ \beta\in\mathfrak{o}_F\right\}.$$

Then, Γ_∞-inequivalence classes in Φ consist of $\phi(x, y)=ax^2+bxy+cy^2$ where a runs over all U_F-classes of totally positive elements in $\mathfrak{o}_F$, i.e., (a) runs over integral ideals, while b runs over all residue classes modulo $(2a)$ satisfying the congruence relation $b^2\equiv\Delta \bmod (4a)$, (under these circumstances, c is uniquely determined by a, b). For an ideal $\mathfrak{a}$ and for $(\Delta)=\mathfrak{f}^2(\Delta_0)$, denote by $r^*_{\Delta_0}(\mathfrak{f}, \mathfrak{a})$ the number of such residue classes, b satisfying the condition above. Then, we obtain

$$(7)\qquad \zeta(s, \Delta)=\sum_{(r)\subset\mathfrak{o}_F}\sum_{\phi\in\Phi/\Gamma_\infty} N_F(\phi(0, r))^{-s} =\zeta_F(2s)\sum_{\mathfrak{a}\subset\mathfrak{o}_F} N_F(\mathfrak{a})^{-s}r^*_{\Delta_0}(\mathfrak{f}, \mathfrak{a}).$$

Therefore, we have only to calculate $r^*_{\Delta_0}(\mathfrak{f}, \mathfrak{a})$, which has a simultaneously multiplicative,

$$r^*_{\Delta_0}(\mathfrak{f}, \mathfrak{a}) = \prod_{\mathfrak{p}} r^*_{\Delta_0}(\mathfrak{p}^{\alpha_{\mathfrak{p}}}, \mathfrak{p}^{\beta_{\mathfrak{p}}}), \qquad \text{if } \mathfrak{f} = \prod_{\mathfrak{p}} \mathfrak{p}^{\alpha_{\mathfrak{p}}},\ \mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{\beta_{\mathfrak{p}}}. \tag{8}$$

Now we investigate $r^*_{\Delta_0}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$ for $\alpha \geqq 0$, $\beta \geqq 0$. When $\mathfrak{p}$ is an odd prime ideal, we have Table I. When $\mathfrak{p}$ is an even prime ideal, the calculations of $r^*_{\Delta_0}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$ are more complicated than in the odd cases. Readjusting them, we have Table II.

Among the results in these Tables, we obtain

$$\begin{aligned} \sum_{\beta=0}^{\infty} r^*_{\Delta_0}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) N_F(\mathfrak{p})^{-s\beta} &= \frac{1+N_F(\mathfrak{p})^{-s}}{1-\chi_\Delta(\mathfrak{p})N_F(\mathfrak{p})^{-s}} \left\{ \frac{1-N_F(\mathfrak{p})^{(\alpha+1)(1-2s)}}{1-N_F(\mathfrak{p})^{1-2s}} \right. \\ &\qquad \left. -\chi_\Delta(\mathfrak{p})N_F(\mathfrak{p})^{-s} \frac{1-N_F(\mathfrak{p})^{\alpha(1-2s)}}{1-N_F(\mathfrak{p})^{1-2s}} \right\} \\ &= \frac{1+N_F(\mathfrak{p})^{-s}}{1-\chi_\Delta(\mathfrak{p})N_F(\mathfrak{p})^{-s}} \sum_{i=0}^{\infty} \mu(\mathfrak{p}^i)\chi_\Delta(\mathfrak{p}^i)N_F(\mathfrak{p}^i)^{-s}\sigma_{1-2s}(\mathfrak{p}^{\alpha-i}). \end{aligned} \tag{9}$$

Then, we get (2) from (7), (8), (9).

Table I

$\nu_{\mathfrak{p}}$	β	$r^*_{\Delta_0}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$
$\nu_{\mathfrak{p}}=0$	$\beta \leqq 2\alpha$	$N_F(\mathfrak{p})^{[\beta/2]}$
$\nu_{\mathfrak{p}}=0$	$\beta > 2\alpha$	$(1+\chi_\Delta(\mathfrak{p}))N_F(\mathfrak{p})^\alpha$
$\nu_{\mathfrak{p}}=1$	$\beta \leqq 2\alpha+1$	$N_F(\mathfrak{p})^{[\beta/2]}$
$\nu_{\mathfrak{p}}=1$	$\beta > 2\alpha+1$	0

Table II

$\nu_{\mathfrak{p}}$	β	$r^*_{\Delta_0}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$
$\nu_{\mathfrak{p}}=0$	$\beta \leqq 2\alpha$	$N_F(\mathfrak{p})^{[\beta/2]}$
$\nu_{\mathfrak{p}}=0$	$\beta > 2\alpha$	$(1+\chi_\Delta(\mathfrak{p}))N_F(\mathfrak{p})^\alpha$
$\nu_{\mathfrak{p}} \geqq 2$	$\beta \leqq 2\alpha+1$	$N_F(\mathfrak{p})^{[\beta/2]}$
$\nu_{\mathfrak{p}} \geqq 2$	$\beta > 2\alpha+1$	0

($[x]$ being Gaussian Symbol.)

Remark. When Δ is exceptional, we use $\alpha'_{\mathfrak{p}}$ instead of $\alpha_{\mathfrak{p}}$ and modified $\chi_\Delta(\mathfrak{p})$ for even $\mathfrak{p}$, to get (6).

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