77. *The Fabry-Ehrenpreis Gap Theorem for Hyperfunctions*

By Takahiro KAWAI
Research Institute for Mathematical Sciences, Kyoto University

(Communicated by Heisuke HIRONAKA, M. J. A., Sept. 12, 1984)

In [7], we have shown that the Fabry-type gap theorems can be most neatly handled by the aid of linear differential equations of infinite order, thus realizing an ideal of Ehrenpreis [3]. Although the classical gap theorems refer to holomorphic functions, it is evident that they are closely related to the analysis of Fourier series on a real domain. The relation is most obvious in the one-dimensional case:

Let $f_+(z)$ (resp., $f_-(z)$) denote $\sum_{n\geqq 0} c_n \exp(ia_n z)$ (resp., $\sum_{n<0} c_n \cdot \exp(ia_n z)$) ($c_n \in \boldsymbol{C}$, $a_n \in \boldsymbol{R}$ and $i=\sqrt{-1}$) and suppose that $f_+(z)$ (resp., $f_-(z)$) determines a holomorphic function on $\{z \in \boldsymbol{C};\ \operatorname{Im} z>0\}$ (resp., $\{z \in \boldsymbol{C};\ \operatorname{Im} z<0\}$). Suppose further that the sequence a_n is sufficiently lacunary so that Theorem 1 of [7] is applicable to them. Let $f(x)$ denote the hyperfunction determined by the pair of holomorphic functions $f_+(z)$ and $f_-(z)$, and suppose that $f(x)$ vanishes near $x=0$. This means, by the definition, that there exists a holomorphic function $F(z)$ defined on $\{z \in \boldsymbol{C};$ either $\operatorname{Im} z \neq 0$ or $|\operatorname{Re} z|<c\ (c>0)\}$ which coincides with $f_\pm(z)$ on $\{z \in \boldsymbol{C};\ \pm \operatorname{Im} z>0\}$, respectively. Then the gap theorem for holomorphic functions entails that both $f_+(z)$ and $f_-(z)$ are holomorphic in a neighborhood of the real axis $\boldsymbol{R}$, and hence their difference $f(x)$ is analytic on $\boldsymbol{R}$. Since $f(x)$ vanishes near $x=0$, this implies that $f(x)$ is identically zero.

In the higher dimensional case, however, such a straightforward connection cannot be observed immediately because of the complexity of the notion of the vanishing of a hyperfunction; it requires a cohomological language. (See [4], Chap. 1, §2, for example.) Still, this trouble due to the higher dimensionality of the problem is only a technical matter, as is usually the case in dealing with hyperfunctions; we can obtain the same result also for the higher dimensional case. This is what we want to report here.

In what follows, for a sequence $a(l)$ ($l \in \boldsymbol{N}=\{0, 1, 2, \cdots\}$) of m-dimensional real vectors, we let $a_j(n)$ ($j=1, \cdots, m$; $n \in \boldsymbol{N}$) denote its j-th reduced sequence in the sense of [7], Definition 1. We also denote $\sum_{j=1}^m |a(l)_j|$ by $|a(l)|$, where $a(l)_j$ denotes the j-th component of $a(l)$.

Theorem. *Let $a(l)$ ($l \in \boldsymbol{N}$) be a sequence of m-dimensional real vectors such that its j-th reduced sequence $a_j(n)$ satisfies the following* *two conditions for some constant* $c>0$:

(1) $\lim_{n\to\infty} n/a_j(n)=0 \qquad (j=1,\cdots,m)$

(2) $|a_j(n)-a_j(n')|\geqq c|n-n'| \qquad (j=1,\cdots,m;\ n,n'\in \boldsymbol{N})$.

Let $c(l)$ $(l\in \boldsymbol{N})$ *be a sequence of complex numbers which satisfies the following condition*:

(3) *For each* $\varepsilon>0$ *there exists a constant* C_ε *for which*

$$|c(l)|\leqq C_\varepsilon \exp(\varepsilon|a(l)|)$$

holds for every l *in* $\boldsymbol{N}$.

Let $f(x)$ $(x\in \boldsymbol{R}^m)$ *denote the hyperfunction* $\sum_{l=0}^{\infty} c(l)\exp(i<a(l),x>)$. *Suppose that* $f(x)$ *vanishes on an open neighborhood of the origin of* $\boldsymbol{R}^m$. *Then* $f(x)$ *vanishes identically.*

Proof. We first note that condition (3) guarantees that the Fourier series $\sum_{l=0}^{\infty} c(l)\exp(i<a(l),x>)$ is a well-defined hyperfunction. (See Proposition 2.4.4 of [4], Chap. 2, for example.) As in [7], let us consider an infinite product $P_j(\partial/\partial x_j)$ of differential operators given by

$$(\partial/\partial x_j)\prod_{n=0}^{\infty}\left(1+\frac{(\partial/\partial x_j)^2}{(a_j(n))^2}\right).$$

We know that conditions (1) and (2) guarantee that $P_j(\partial/\partial x_j)$ $(j=1,\cdots,m)$ is a well-defined linear differential operator of infinite order. Further, the hyperfunction $f(x)$ solves the following system $\mathcal{M}$ on $\boldsymbol{R}^m$:

$$\mathcal{M}: P_j(\partial/\partial x_j)f(x)=0, \qquad j=1,\cdots,m.$$

Using an invertibility theorem for linear differential operators of infinite order ([2], Theorem 1. See also [6] and [1]), we find $\mathrm{Ch}(\mathcal{M})$, the characteristic set of $\mathcal{M}$ (in the sense of [8]), is

$$\{(z,\zeta)\in \boldsymbol{C}^m\times \boldsymbol{C}^m\cong T^*\boldsymbol{C}^m;\ \zeta_j\ (j=1,\cdots,m) \text{ is pure imaginary}\}.$$

Hence the boundary of $\Omega \underset{\text{def}}{=} \{(x,\sqrt{-1}\xi)\in T^*_{\boldsymbol{R}^m}\boldsymbol{C}^m,\ \sum_{j=1}^m x_j^2<t\}$ is microhyperbolic with respect to $\mathcal{M}$ for any $t>0$. Hence, by using Theorem 5.1.2 of [5] on the propagation of analyticity for solutions of microhyperbolic systems, we find that the hyperfunction $f(x)$, which is zero, and hence analytic, near the origin, is analytic all over $\boldsymbol{R}^m$. Since it is zero near the origin, it identically vanishes. Q.E.D.

Remark. Suppose the same conditions on $a(l)$ and $c(l)$ as in the theorem. If we suppose that $f(x)$ is analytic near the origin, instead of supposing that $f(x)$ is zero near the origin, then, by the same reasoning as above, we find that $f(x)$ is analytic all over $\boldsymbol{R}^m$. This fact might be more akin to the classical gap theorems in its nature.

References

[1] Aoki, T.: The exponential calculus of microdifferential operators of infinite order. V. Proc. Japan Acad., **60A**, 8–9 (1984).

[2] Aoki, T., M. Kashiwara and T. Kawai: On a class of linear differential operators of infinite order with finite index (to appear).
[3] Ehrenpreis, L.: Fourier Analysis in Several Complex Variables. Wiley-Interscience, New York-London-Sydney-Toronto (1970).
[4] Kashiwara, M., T. Kawai and T. Kimura: Foundations of Algebraic Analysis. Kinokuniya, Tokyo (1980) (in Japanese). (The English edition will be published by Princeton University Press).
[5] Kashiwara, M. and P. Schapira: Micro-hyperbolic systems. Acta Math., **142**, 1–55 (1979).
[6] Kawai, T.: On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients. J. Fac. Sci. Univ. Tokyo, Sect. IA, **17**, 467–517 (1970).
[7] ——: The Fabry-Ehrenpreis gap theorem and linear differential equations of infinite order (to appear). (RIMS preprint No. 484.)
[8] Sato, M., M. Kashiwara and T. Kawai: Microlocal analysis of theta functions (to appear in Adv. Studies in Pure Math., **4** (1984)).