76. *On Totally Multiplicative Signatures of Natural Numbers*

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1. Introduction. Let $\mathbf{N}$ be the set of all natural numbers and σ a mapping from $\mathbf{N}$ to the set $\{\pm 1\}$ satisfying the condition $\sigma(ab) = \sigma(a)\sigma(b)$ for all $a, b \in \mathbf{N}$. We call such a mapping σ *a totally multiplicative signature*. We have $\sigma(a^2)=1$, particularly $\sigma(1)=1$. The constant signature $\sigma(a)=1$ for all $a \in \mathbf{N}$ is called *trivial*. In the following, we are concerned with non-trivial totally multiplicative signatures, called simply signatures and denoted by σ. Let $\Pi(\sigma)$ be the set of all primes p, for which $\sigma(p)=-1$. σ is obviously determined by $\Pi(\sigma)$. When $\Pi(\sigma)$ coincides with the set of all primes, then σ is Liouville's function λ. S. Chowla conjectured that, given any finite sequence $\varepsilon_1, \cdots, \varepsilon_g$, $\varepsilon_m = \pm 1$, then $\lambda(x+m)=\varepsilon_m (1 \leqslant m \leqslant g)$ will have infinitely many solutions (cf. [1], [5]). In [4], I. Schur and G. Schur proved that the followings are the only signatures for which $\sigma(x)=\sigma(x+1)=\sigma(x+2)=1$ does not occur.

I. If $\sigma(3)=1$, then $\sigma(3n+1)=1$, $\sigma(3n+2)=-1$, $\sigma(3^k t)=\sigma(t)$ for all n, k, t with $(t, 3)=1$.

II. If $\sigma(3)=-1$, then $\sigma(3n+1)=1$, $\sigma(3n+2)=-1$, $\sigma(3^k t) =(-1)^k \sigma(t)$ for all n, k, t with $(t, 3)=1$.

Furthermore they proved that $\sigma(x)=1$, $\sigma(x+1)=-1$, $\sigma(x+2)=1$ has always a solution for any σ.

In this paper we prove the following theorem.

Theorem. *Let σ be a totally multiplicative signature for which $\Pi(\sigma)$ contains at least two primes. Then*

(i) $\sigma(x)=-1$, $\sigma(x+1)=-1$ *has infinitely many solutions,*

(ii) $\sigma(x)=-1$, $\sigma(x+1)=1$, $\sigma(x+2)=-1$ *has a solution and if $\sigma(2)=1$, it has infinitely many solutions.*

Our result contains a special case of Chowla's conjecture.

Henceforth we simply write either $(n)_+$ or $(n)_-$ instead of $\sigma(n)=1$ or $\sigma(n)=-1$, respectively.

2. Proof of Theorem. Let p, q be the smallest and the next smallest elements of $\Pi(\sigma)$. Then we have $1<p<q$, $(p, q)=1$.

Proof of (i). The congruence $qx \equiv 1 \pmod p$ has a unique solution x_0 in the interval $1 \leqslant x \leqslant p-1$. So there exists $r \in \mathbf{N}$ such that $qx_0 = pr+1$. Similarly the congruence $qy \equiv -1 \pmod p$ has a unique solution y_0 in the interval $1 \leqslant y \leqslant p-1$ and we have $s \in \mathbf{N}$ such that $qy_0 = ps-1$.

We now consider two pairs of two consecutive natural numbers $qx_0-1=pr$, qx_0; qy_0, $qy_0+1=ps$. As $1 \leqslant x_0$, $y_0 \leqslant p-1$, it is easy to see that $1 \leqslant r < q$, $1 \leqslant s < q$. Therefore, from the definition of p and q, if $p \nmid r$, we have $(r)_+$ and if $p \nmid s$, we obtain $(s)_+$. So according as either $p \nmid r$ or $p \nmid s$, we have either $(qx_0-1)_-$, $(qx_0)_-$ or $(qy_0)_-$, $(qy_0+1)_-$, respectively. Therefore we have only to show that at least one of the two numbers r, s is not divisible by p.

Suppose both $p \mid r$ and $p \mid s$. By the equalities $qx_0 = pr+1$, $qy_0 = ps-1$, we have $q(x_0+y_0) = p(r+s)$. By our assumption, we have $p \mid (r+s)$ and so $p^2 \mid q(x_0+y_0)$. But, as $(p, q)=1$, we have $p^2 \mid (x_0+y_0)$. This contradicts the fact that $2 \leqslant x_0+y_0 \leqslant 2p-2 < p^2$. This assures an existence of a natural number m with $(m)_-$, $(m+1)_-$.

From the above proof we see that at least one of the linear equations $px-qy=\pm 1$ has a solution $x=u$, $y=v$ such that $(pu)_-$, $(qv)_-$, $1 \leqslant u \leqslant q-1$, $1 \leqslant v \leqslant p-1$, and $p \nmid u$.

We consider the diophantine equation $(pu)x^2-(qv)y^2=\pm 1$, where the sign corresponds to the linear equation which has the solution $x=u$, $y=v$.

The above quadratic equation has integral coefficients and an integral solution $x=y=1$. Its discriminant d is equal to $4pquv$. So d is positive and is not a square number since $q^2 \nmid d$. Therefore this equation has infinitely many integral solutions (cf. [3], p. 150, Th. 8–10), and we have $(pux^2)_-$, $(qvy^2)_-$.

Proof of (ii). Case $p=2$. First we consider the case $(3)_+$. Then $(1)_+$, $(2)_-$, $(3)_+$. Therefore if $(7)_+$, we obtain $(6)_-$, $(7)_+$, $(8)_-$. So suppose $(7)_-$. If $(5)_+$, we have $(8)_-$, $(9)_+$, $(10)_-$. So suppose moreover $(5)_-$. If $(13)_-$, then $(13)_-$, $(14)_+$, $(15)_-$. Therefore suppose again $(13)_+$. Then we obtain $(24)_-$, $(25)_+$, $(26)_-$. This assures that any signed sequence $(1)_+$, $(2)_-$, $(3)_+$, $\cdots$ of natural numbers contains a triple $-$, $+$, $-$ of consecutive signs.

Similarly we can also prove that there exists a natural number n such that $(n)_-$, $(n+1)_+$, $(n+2)_-$ in the case $(3)_-$.

These prove (ii) of Theorem in the case $p=2$.

Case $p>2$. Then $(2)_+$. From now on we assume that there are no three consecutive natural numbers n, $n+1$, $n+2$ such that $(n)_-$, $(n+1)_+$, $(n+2)_-$.

By the part (i) of Theorem, there exist infinitely many natural numbers m which satisfy $(m)_-$, $(m+1)_-$. So for each m, as $(2)_+$, we have $(2m)_-$, $(2(m+1))_-$. Therefore by the above assumption, we have $(2m)_-$, $(2m+1)_-$, $(2(\mathrm{m}+1))_-$. Repeating this process $2k$ times, we can find $2^{2k}+1$ consecutive natural numbers $2^{2k}m$, $2^{2k}m+1, \cdots, 2^{2k}(m+1)$ such that their σ-values are -1. So we will obtain a contradiction if we can find a square number among these $2^{2k}+1$ numbers for a sufficiently large k.

Consider the interval $[2^k\sqrt{m}, 2^k\sqrt{m+1}]$. If k is sufficiently large, we obtain $2^k\sqrt{m+1}-2^k\sqrt{m}>1$. So there exists a natural number h such that $2^k\sqrt{m}<h<2^k\sqrt{m+1}$. So we have $2^{2k}m<h^2<2^{2k}(m+1)$. This is a contradiction. Therefore for each m with $(m)_-$, $(m+1)_-$, we can find a natural number g with $(g)_-$, $(g+1)_+$, $(g+2)_-$, which proves (ii).

Remark. It seems difficult to find necessary and sufficient conditions which assure the existence of a natural number m with $(m)_-$, $(m+1)_-$, $(m+2)_-$.

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References

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