75. *A Family of Finite Nilpotent Groups*

By Katsuya MIYAKE
Department of Mathematics, College of General Education, Nagoya University

(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1984)

1. Introduction. The primary purpose of this paper is to show that Theorems 1 and 2 in our previous work [3] can be extended to a much wider class of p-genera of capitulation than of regular ones, as was mentioned there in Remark 3. But we shall be concerned here thoroughly with finite nilpotent groups.

As far as transfers of a p-group G to its normal subgroups are concerned, it was confirmed in [2] that we have

$$V_{G\to N}(g)=g^{[G:N]}\cdot[N, N]$$

for every $g\in G$ and every normal subgroup N of G if G is regular. Here $V_{G\to N}$ is the transfer of G to N and $[N, N]$ denotes the commutator subgroup of N. In this paper, we show that this phenomenon on transfers appears in the nilpotent groups of a wider family than that of regular groups. In fact, this new family is closed under the operation of taking direct products though the direct product of two regular p-groups is not necessarily regular in general (e.g. Weichsel [4]). It is also closed under the operation of taking quotient groups. But it should be noted that it is not closed under taking (normal) subgroups. We shall give a method of constructing members of the new family from a special type of p-groups which do not belong to the family, and see that there are a lot of irregular p-groups in the family even if $p=2$.

2. The property TNP of finite nilpotent groups. Let G be a finite nilpotent group.

Definition. G has the property TNP, or is a TNP-group if the *transfer* of G to every *normal* subgroup N of G coincides with the $[G:N]$-th *power* map modulo $[N, N]$, or in other words, if we have

$$V_{G\to N}(g)=g^{[G:N]}\cdot[N, N] \qquad \text{for } \forall g\in G$$

for every normal subgroup N of G.

Proposition 1. *A quotient group of a TNP-group is a TNP-group.*

Proof. Let G be a TNP-group, and M be a normal subgroup of G. Put $\bar{G}=G/M$. Then every normal subgroup $\bar{N}$ of $\bar{G}$ corresponds to a normal subgroup N of G containing M. Then $N\backslash G$ and $\bar{N}\backslash\bar{G}$ are canonically isomorphic. Therefore, by the definition of transfers, we have the commutative diagram,

$$\begin{array}{ccc} G & \xrightarrow{V_{G\to N}} & N/[N,N] \\ \pi\downarrow & & \downarrow\pi' \\ \bar{G}=G/M & \xrightarrow[V_{\bar{G}\to\bar{N}}]{} & \bar{N}/[\bar{N},\bar{N}] \end{array}$$

where π is the natural projection and π' is the homomorphism induced from π. The proposition is clear from the diagram.

Theorem 1. *Every regular p-group has the property TNP.*

Proof. Let G be a regular p-group and N be an arbitrary normal subgroup of G. Then $[N, N]$ is normal in G. Put $M=[N, N]$ and let us use the commutative diagram in the proof of Proposition 1. Then π' is the identity of $\bar{N}=N/[N, N]$. Since $\bar{G}=G/M$ is also a regular p-group, we can apply Theorem 4 of [2, I-1] to $\bar{G}$ and its normal abelian subgroup $\bar{N}$, and obtain the theorem at once.

3. The construction of TNP-groups. We show a method of constructing a TNP-group using not only TNP-groups but also groups without the property TNP. First, we show

Proposition 2. *Let G be a finite nilpotent group, and M and X be its normal subgroups. Suppose that G/M has the property TNP. Then for each $g\in G$, we have*

$$V_{G\to X}(g)\equiv g^{[G:X]}\cdot[X,X] \bmod V_{XM\to X}(M\cap[G,XM])\cdot[X,M].$$

Proof. For $g\in G$, take $t\in G$ so that $V_{G\to XM}(g)=t\cdot[XM,XM]$. Then $V_{G\to X}(g)=V_{XM\to X}(t)$ by Huppert [1, Ch. IV, 1.6]. Put $y=g^{-[G:XM]}\cdot t$. This is an element of $M\cdot[XM,XM]=M\cdot[X,X]$ because G/M is a TNP-group by the assumption. It is clear by [1, Ch. IV, 1.7] that y belongs to $[G,XM]$. Therefore we have $y\in M\cdot[X,X]\cap[G,XM]=(M\cap[G,XM])\cdot[X,X]$. Replacing t by $t\cdot u$ with $u\in[X,X]$ if necessary, we may assume that $V_{G\to XM}(g)=t\cdot[XM,XM]$ and $y=g^{-[G:XM]}\cdot t\in M\cap[G,XM]$. Then we have

$$\begin{aligned} V_{G\to X}(g)&=V_{XM\to X}(t)=V_{XM\to X}(g^{[G:XM]})\cdot V_{XM\to X}(y)\\ &\equiv(g^{[G:XM]})^{[XM:X]}\cdot V_{XM\to X}(y) \bmod [XM,X]\\ &\equiv g^{[G:X]}\cdot V_{XM\to X}(y) \bmod [XM,X]. \end{aligned}$$

Since $[XM,X]=[X,M]\cdot[X,X]$, we have the desired result.

Corollary. *Let G be a p-group, and $K_1(G)=G\supset K_2(G)\supset\cdots\supset K_n(G)\supset\cdots$ be the lower central series of G. Let X be a normal subgroup of G. Then for each $g\in G$, we have*

$$V_{G\to X}(g)\equiv g^{[G:X]}\cdot[X,X] \bmod V_{XK_p(G)\to X}(K_p(G)\cap[G,X]K_{p+1}(G))\cdot[X,K_p(G)].$$

Proof. Since the class of $G/K_p(G)$ is less than p, it is a regular p-group (see Huppert [1, Ch. III, 10.2 a)]). The corollary follows from Theorem 1 and Proposition 2 at once if we take $M=K_p(G)$.

Theorem 2. *Let G and H be finite nilpotent groups, and M and N be normal subgroups of G and H, respectively. Suppose that G/M and H/N have the property TNP. Then the quotient group $(G\times H)/D$*

of the direct product of G and H is a TNP-group if the normal subgroup D satisfies the following conditions (1) *and* (2):

(1) $M\cap[G, G]\subset D\cdot[H, H]$; (2) $N\cap[H, H]\subset D\cdot[G, G]$.

Here G and H are considered naturally embedded in $G\times H$.

Proof. Put $S=G\times H$ and $\bar{S}=S/D$, and let $\pi: S\to\bar{S}$ be the natural projection. Every normal subgroup $\bar{U}$ of $\bar{S}$ corresponds to a normal subgroup U of S containing D by π. Let $V=V_{S\to U}$ and $\bar{V}=V_{\bar{S}\to\bar{U}}$ be the transfers. As is in the proof of Proposition 1 in § 2, we have $\bar{V}\circ\pi=\pi'\circ V$ where $\pi': U/[U, U]\to\bar{U}/[\bar{U}, \bar{U}]$ is the homomorphism induced from π. Therefore, it is sufficient to show that

$$(*) \qquad V(a)\equiv a^{[S:U]}\cdot[U, U] \bmod D\cdot[U, U] \qquad \text{for } \forall a\in S.$$

It is enough to show $(*)$ for each $g\in G$ and for each $h\in H$. In fact, if it be done, then for $a=g\cdot h\in S$ with $g\in G$ and $h\in H$, we have

$$V(a)=V(gh)=V(g)\cdot V(h)\equiv g^{[S:U]}\cdot h^{[S:U]}\cdot[U, U] \bmod D\cdot[U, U].$$

Since g and h commute with each other, we have

$$g^{[S:U]}\cdot h^{[S:U]}=(gh)^{[S:U]}=a^{[S:U]}.$$

Now, we show $(*)$ for $a=g\in G$. Put $T=U\cdot H$ and $X=G\cap T$. Then T and X are normal in S. Since a set of representatives for $X\backslash G$ is also that of $T\backslash S$, we see $V_{S\to T}(g)=V_{G\to X}(g)\cdot[T, T]$ by the definition. Since G/M is a TNP-group, we can find, by Proposition 2, an element u of $M\cap[G, G]$ such that $V_{G\to X}(g)=g^{[G:X]}\cdot u\cdot[X, X]$. (Note that $V_{XM\to X}(M\cap[G, XM])\cdot[X, M]\subset M\cap[G, G]$ because X and M are normal in G.) Then by Huppert [1, Ch. IV, 1.6], we have

$$V_{S\to U}(g)=V_{T\to U}(g^{[G:X]})\cdot V_{T\to U}(u).$$

Put $x=g^{[G:X]}$. It commutes with every element of H. Since $T=U\cdot H$, we can choose a set of representatives of $U\backslash T$ from H. Then it is easy to see that each $\langle x\rangle$-orbit in $U\backslash T$ consists of $[\langle x\rangle U: U]$ cosets, and that $V_{T\to U}(x)=x^{[T:U]}\cdot[U, U]$ by [1, Ch. IV, 1.7]. Since $[G:X]=[S:T]$, we have $V_{T\to U}(g^{[G:X]})=(g^{[G:X]})^{[T:U]}\cdot[U, U]=g^{[S:U]}\cdot[U, U]$. As for $V_{T\to U}(u)$, we can find $d\in D$ and $e\in[H, H]$ such that $u=d\cdot e$ by the assumption (1). Since $T=UH$, e belongs to $[T, T]$. Therefore $V_{T\to U}(u)=V_{T\to U}(d)$. Because D is normal in T, we have $V_{T\to U}(d)\in D\cdot[U, U]/[U, U]$ by [1, Ch. IV, 1.7]. Thus we have shown $(*)$ for $a=g\in G$. For $a=h\in H$, we can similarly show $(*)$ replacing the roles of G and H in the above argument by each other and the condition (1) by (2). Then the proof of the theorem is completed.

As the special case where $M=N=D=1$, we have

Corollary 1. *The direct product of two TNP-groups is also a TNP-group.*

Corollary 2. *Let G be a finite nilpotent group, and M be a normal subgroup of G. Suppose that G/M is a TNP-group, and that $M\cap[G, G]$ lies in the center $Z(G)$ of G. Let* Inn(G) *be the group of* *all the inner automorphisms of G. Then the semi-direct product* $\mathrm{Inn}(G)\cdot G$ *has the property TNP.*

Proof. Take a copy H of G and fix an isomorphism $\iota: G \to H$. Put $S = G \times H$ and $D = \{(g, \iota(g)) \mid g \in Z(G)\}$. Then D is a normal subgroup of S. We consider G and H embedded in S. Therefore, for example, $(g, \iota(g)) = g \cdot \iota(g)$. It is easy to see that the conditions (1) and (2) of Theorem 2 are satisfied if we take $N = \iota(M)$. Therefore $\bar{S} = S/D$ is a TNP-group. Put $\tilde{G} = \{g \cdot \iota(g) \mid g \in G\}$. Since $g \in G$ and $\iota(g') \in H$ commute each other in S, $\tilde{G}$ is a subgroup of S and contains D as its center $Z(\tilde{G})$. Furthermore, S is the semi-direct product of $\tilde{G}$ and G where $\tilde{G}$ acts on G through inner automorphisms of S. It is now clear that $\bar{S} = S/D = (\tilde{G}/D)\cdot G$ is isomorphic to the semi-direct product $\mathrm{Inn}(G)\cdot G$. Hence this is a TNP-group by Theorem 2. Q.E.D.

Corollary 3. *Let G be a finite nilpotent group. If either one of the following conditions* (a) *and* (b) *is satisfied, then* $\mathrm{Inn}(G)\cdot G$ *is a TNP-group*:

(a) *$G/Z(G)$ is a TNP-group*;

(b) *G is a p-group, the class of which is less than or equal to p.*

Proof. On either case, we can take $M = Z(G)$ to apply Corollary 2 because a p-group of class less than p is regular and a TNP-group.

Remark. For each prime p, there is a p-group G of class p which does not have the property TNP. (Cf. [1, Ch. III, 10.15] and [2, II-3, 6].) But $\mathrm{Inn}(G)\cdot G$ is a TNP-group by Corollary 3. This shows that the family of TNP-groups is not closed under the operation of taking (normal) subgroups. It seems very interesting to find a p-group which cannot be a (normal) subgroup of any TNP-groups.

Finally, we state an immediate consequence of Proposition 1 and Corollary 1 to Theorem 2.

Theorem 3. *A finite nilpotent group has the property TNP if and only if every Sylow subgroup is a TNP-group.*

References

[1] B. Huppert: Endliche Gruppen I. Springer-Verlag, Berlin · Heidelberg · New York (1967).

[2] K. Miyake: The application of the Principal Ideal Theorem to p-groups. Preprint series, 1984, No. 3, Dept. of Math., College of General Education, Nagoya Univ. (1984).

[3] ——: On capitulation of ideals of an algebraic number field. Proc. Japan Acad., **60**A, 232–235 (1984).

[4] P. Weichsel: Regular p-groups and varieties. Math. Zeitschr., **95**, 223–231 (1967).