73. *On Characters of Irreducible Highest Weight Representations of Witt Algebra*

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1. Introduction. Many results on characters of irreducible highest weight representations of Witt algebra were obtained by several authors (V. G. Kac [3], [4] and A. Rocha-Caridi and N. R. Wallach [6]). In this paper we determine the remaining characters by using the methods of [6].

The *Witt algebra* is an infinite dimensional complex Lie algebra with basis $\{E_i\}_{i\in Z}$ which have the following commutation relations:

$$[E_i, E_j]=(j-i)E_{i+j} \qquad i, j \in Z.$$

It is also known as a Lie algebra of polynomial vector fields on the circle. Let us denote the Witt algebra by $\mathfrak{g}$.

A *highest weight module* of $\mathfrak{g}$ is defined as follows.

Definition. A $\mathfrak{g}$-module M is called the *highest weight module* with highest weight $\lambda \in C$ if there exists a nonzero vector v such that

(1) $E_i \cdot v=0$ for $i>0$

(2) $E_0 \cdot v=\lambda v$

(3) M is generated by v as $\mathfrak{g}$-module.

If M is a highest weight module with highest weight λ, then M is decomposed as a direct sum of its weight spaces relative to the action of E_0:

$$M=\oplus_{i=0}^{\infty} M_{\lambda-i}$$

where $M_{\lambda-i}=\{u \in M ; E_0 \cdot u=(\lambda-i)u\}$.

We define the formal character of M by

$$\mathrm{ch}\, M=\textstyle\sum_{\nu\in C} (\dim M_{-\nu})e^{\nu}$$

where e^{ν} is a formal exponential.

For any complex number λ there exists a unique irreducible highest weight module $L(\lambda)$ with highest weight λ.

Our main theorem is the following.

Theorem. *Put* $\lambda_m=-(m^2-1)/24$ *for nonnegative integer* m.

(a) *For* $\lambda=\lambda_m$, $m\equiv 2 \pmod 6$, *we have*

$$\mathrm{ch}\, L(\lambda)=e^{-\lambda}\phi(e)^{-1}(1-e^{2(m+4)/3}).$$

(b) *For* $\lambda=\lambda_m$, $m\equiv 4 \pmod 6$, *we have*

$$\mathrm{ch}\, L(\lambda)=e^{-\lambda}\phi(e)^{-1}(1-e^{(m+2)/3}).$$

where $\phi(e)=\prod_{i=1}^{\infty}(1-e^i)$ *is the generating function of the classical partition function.*

2. Preliminaries. In this section we state known results about Verma modules, contravariant form and characters of highest weight modules.

We set $\mathfrak{h}=\boldsymbol{C}E_0$, $\mathfrak{n}=\oplus_{i>0}\boldsymbol{C}E_i$ and $\mathfrak{b}=\mathfrak{n}+\mathfrak{h}$. For a complex number λ, let $\boldsymbol{C}(\lambda)$ be the one dimensional $\mathfrak{b}$-module where $\mathfrak{n}$ acts trivially and E_0 acts by scalar multiple λ. We denote by $M(\lambda)$ the universal highest weight module with highest weight $\lambda: M(\lambda)=U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\boldsymbol{C}(\lambda)$, where $U(\ )$ is the universal enveloping algebra. $M(\lambda)$ is called Verma module. It is easy to see that the character of $M(\lambda)$ is given by

$$\operatorname{ch} M(\lambda)=e^{-\lambda}\sum_{n=0}^{\infty}p(n)e^{n}=e^{-\lambda}\phi(e)^{-1}$$

where p is the classical partition function. Let $L(\lambda)$ denote the unique irreducible quotient of $M(\lambda)$.

Now we describe the structures of Verma modules using contravariant forms. Let σ be the involutive antiautomorphism of $\mathfrak{g}$ such that $\sigma(E_i)=E_{-i}$ for all $i\in\boldsymbol{Z}$. Let σ also denote the extension of σ to be an involutive antiautomorphism of $U(\mathfrak{g})$. Let M be a $\mathfrak{g}$-module. A symmetric bilinear form $(\ ,\ )$ on M is contravariant (relative to σ) if $(X\cdot u, w)=(u,\sigma(X)\cdot w)$ for all $X\in U(\mathfrak{g})$, $u, w\in M$. The following Proposition can be proved with the same type of arguments used in Jantzen ([2]).

Proposition 1. (a) *Let M be a $\mathfrak{g}$-module and let $(\ ,\ )$ be a contravariant form. Then $(M_\mu, M_\lambda)=0$ if $\mu\neq\nu$.*

(b) *Let M be a highest weight module. Then there exists a nonzero contravariant form $(\ ,\ )$ which is unique up to nonzero scalar multiple and* $\operatorname{Rad}(\ ,\ )=\{u\in M\,;\,(u,w)=0$ *for all $w\in M\}$ is the proper maximal submodule of M.*

In particular, by Proposition 1, there exists a nonzero contravariant form on Verma module $M(\lambda)$. We fix such a contravariant form and denote it by $(\ ,\ )_\lambda$. We denote by $(\ ,\ )_{\lambda-m}$ the restriction of $(\ ,\ )_\lambda$ to $M(\lambda)_{\lambda-m}$.

The following Theorem is due to Kac ([3]).

Theorem 2. *In above notations we have*

$$\det(\ ,\ )_{\lambda-m}=\text{Const.}\prod_{i=1}^{m}\prod_{r|i}\{\lambda+([3r-2i/r]^2-1)\}^{p(m-i)}$$

where constant depends on the choice of basis of $M(\lambda)_{\lambda-i}$.

Next Proposition provides the structure of $M(\lambda)$.

Proposition 3 (Rocha-Caridi and Wallach [6]). *Let λ be a complex number. Then there exists a filtration of $\mathfrak{g}$-submodule of $M(\lambda)$,*

$$M(\lambda)=M(\lambda)_0\supseteqq M(\lambda)_1\supseteqq M(\lambda)_2\supseteqq\cdots$$

such that

(a) *$M(\lambda)_1$ is the proper maximal submodule of $M(\lambda)$*

(b) *for every $k\geqq 0$ there exists a nondegenerate contravariant form on $M(\lambda)_k/M(\lambda)_{k+1}$ if $M(\lambda)_k\neq M(\lambda)_{k+1}$*

(c) $\sum_{i=1}^{\infty}\operatorname{ch} M(\lambda)_i=e^{-\lambda}\sum_{i=1}^{\infty}\operatorname{ord}_{t=0}(\det(\ ,\ )_{\lambda+t-i})e^{i}$

where $\mathrm{ord}_{t=0}(\det(\ ,\ )_{\lambda+t-i})$ *is the order of zero of the polynomial* $\det(\ ,\ )_{\lambda+t-i}$ *at* $t=0$.

3. Proof of the main Theorem. First, we apply Proposition 3 to our case and write down the character sum formula of Proposition 3 (c). As in the preceeding sections we put $\lambda_m=-(m^2-1)/24$ for non-negative integer m.

Lemma. *Let* $M(\lambda)=M(\lambda)_0\supseteqq M(\lambda)_1\supseteqq M(\lambda_2)\supseteqq\cdots$ *be the filtration of Proposition* 3.

(a) *If* $\lambda=\lambda_m$, $m\equiv 2 \pmod 6$, *then*

$$\sum_{i=1}^{\infty}\mathrm{ch}\,M(\lambda)_i=\sum_{k=1}^{\infty}\{\mathrm{ch}\,M(\lambda_{m+12k-4})+\mathrm{ch}\,M(\lambda_{m+12k})\}. \tag{1}$$

(b) *If* $\lambda=\lambda_m$, $m\equiv 4 \pmod 6$, *then*

$$\sum_{i=1}^{\infty}\mathrm{ch}\,M(\lambda)_i=\sum_{k=1}^{\infty}\{\mathrm{ch}\,M(\lambda_{m+12k-8})+\mathrm{ch}\,M(\lambda_{m+12k})\}. \tag{2}$$

Proof. (a) For $m=6n+2$, $n\in \boldsymbol{Z}$, the solutions of

$$\lambda_m+([3r-r/2i]^2-1)/24=0 \qquad i\in \boldsymbol{Z},\ r\,|\,i$$

are given by

$$\begin{cases} r=2(k+n) \\ i=2(k+n)(3k-1)\end{cases} \quad \begin{cases} r=2k \\ i=2k(3n+3k+1)\end{cases} \quad k=1,2,3,\cdots.$$

It is easy to see that $\lambda_m+2(n+k)(3k-1)=\lambda_{m+12k-4}$, and $\lambda_m+2k(3n+3k+1)=\lambda_{m+12k}$. Then (a) is a consequence of simple calculations using Proposition 3 and Theorem 1.

(b) can be proved by the same way.

Using the above Lemma, we now prove the following Theorem which implies the Theorem in Introduction.

Theorem. (a) *If* $\lambda=\lambda_m$, $m\equiv 2 \pmod 6$, *then the proper maximal* $\mathfrak{g}$*-submodule of* $M(\lambda)$ *is isomorphic to* $M(\lambda_{m+8})$:

$$L(\lambda)\simeq M(\lambda)/M(\lambda_{m+8}) \qquad \mathrm{ch}\,L(\lambda)=\mathrm{ch}\,M(\lambda)-\mathrm{ch}\,M(\lambda_{m+8}).$$

(b) *If* $\lambda=\lambda_m$, $m\equiv 4 \pmod 6$, *then the proper maximal* $\mathfrak{g}$*-submodule of* $M(\lambda)$ *is isomorphic to* $M(\lambda_{m+4})$:

$$L(\lambda)\simeq M(\lambda)/M(\lambda_{m+4}) \qquad \mathrm{ch}\,L(\lambda)=\mathrm{ch}\,M(\lambda)-\mathrm{ch}\,M(\lambda_{m+4}).$$

Proof. Let

$$M(\lambda)=M(\lambda)_0\supseteqq M(\lambda)_1\supseteqq M(\lambda_2)\supseteqq\cdots$$

be the filtration of $\mathfrak{g}$-submodules determined by Proposition 3. Then $M(\lambda)_1$ is the proper maximal $\mathfrak{g}$-submodule of $M(\lambda)$.

If $\lambda=\lambda_m$, $m\equiv 2 \pmod 6$, then by Lemma (a) there exists a nonzero weight vector $u\in M(\lambda)_1$ of weight λ_{m+8}. By (1) u is unique up to nonzero scalar multiple and λ_{m+8} is a maximal weight of $M(\lambda)_1$ (i.e. if $\lambda_{m+8}+n$, $n\geqq 0$ is a weight of $M(\lambda)_1$, then $n=0$). We denote by $N(\lambda)_1$ the $\mathfrak{g}$-module generated by u. $N(\lambda)_1$ is isomorphic to $M(\lambda_{m+8})$ and contained in $M(\lambda)_1$.

If $\lambda=\lambda_m$, $m\equiv 4 \pmod 6$, then using Lemma (b) and the same arguments above we obtain a $\mathfrak{g}$-submodule $N'(\lambda)_1$ of $M(\lambda)_1$ which is isomorphic to $M(\lambda_{m+4})$.

We apply the same arguments to $N(\lambda)_1$ and $N'(\lambda)_1$ and continue these processes. Thus we obtain the following filtrations of $\mathfrak{g}$-modules.

(A) For $\lambda=\lambda_m$, $m\equiv 2 \pmod 6$

$$M(\lambda)=N(\lambda)_0\supseteqq N(\lambda)_1\supseteqq N(\lambda)_2\supseteqq\cdots$$

where $N(\lambda)_k\cong M(\lambda_{m+6k+2})$ if k is odd and $N(\lambda)_k\cong M(\lambda_{m+6k})$ if k is even.

(B) For $\lambda=\lambda_m$, $m\equiv 4 \pmod 6$

$$M(\lambda)=N'(\lambda)_0\supseteqq N'(\lambda)_1\supseteqq N'(\lambda)_2\supseteqq\cdots$$

where $N'(\lambda)_k\cong M(\lambda_{m+6k-2})$ if k is odd and $N'(\lambda)_k\cong M(\lambda_{m+6k})$ if k is even.

Obviously we have

$$\sum_{i=1}^{\infty}\mathrm{ch}\,N(\lambda)_i=\sum_{k=1}^{\infty}\{\mathrm{ch}\,M(\lambda_{m+12k-4})+\mathrm{ch}\,M(\lambda_{m+12k})\} \tag{3}$$

$$\sum_{i=1}^{\infty}\mathrm{ch}\,N'(\lambda)_i=\sum_{k=1}^{\infty}\{\mathrm{ch}\,M(\lambda_{m+12k-8})+\mathrm{ch}\,M(\lambda_{m+12k})\}. \tag{4}$$

By Proposition 3 (b) and above constructions of filtrations of $\mathfrak{g}$-modules, we conclude that $N(\lambda)_i$ (or $N'(\lambda)_i$) must be contained in $M(\lambda)_i$. Then by formulas (1)–(4) $N(\lambda)_i$ (or $N'(\lambda)_i$) are equal to $M(\lambda)_i$ for all $i\geqq 0$. In particular $N(\lambda)_1$ (or $N'(\lambda)_1$) is equal to $M(\lambda)_1$. Hence we have proved our Theorem.

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Note added in proof. The same results were obtained by A. Rocha-Caridi and N. R. Wallach (Math. Zeit. 185 (1984)).

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