71. *Teichmüller Spaces of Seifert Fibered Manifolds with Infinite π_1*

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(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1984)

It is known that geometric structure which 3-manifolds can possess is one of H^3, E^3, S^3, $H^2\times \boldsymbol{R}$, $S^2\times \boldsymbol{R}$, $\widetilde{SL}_2$, Nil, Sol, ([9]). Teichmüller space of a geometric manifold M is the set of all metric (of the geometry) on M factored by isotopy. The topology is the quotient of C^∞-topology. For H^3, if M is a Haken 3-manifold, the Teichmüller space is trivial by Mostow's rigidity theorem. In this note we determine Teichmüller spaces of geometric 3-manifolds modelled on $H^2\times \boldsymbol{R}$, $\widetilde{SL}_2$, E^3, Nil, $S^2\times \boldsymbol{R}$. We denote the Teichmüller space of M by $\mathcal{T}(M)$. Throughout this note M is compact and orientable.

§1. Teichmüller spaces of 2-orbifolds. As geometric manifolds modelled on $H^2\times \boldsymbol{R}$, $\widetilde{SL}_2$, E^3, Nil, $S^2\times \boldsymbol{R}$ are Seifert fibered manifolds, we consider Teichmüller spaces of base orbifolds first.

Theorem 1. *Let O be a compact hyperbolic 2-orbifold (possibly nonorientable with geodesic boundaries) with k cone points and without other singularities. Then $\mathcal{T}(O)\cong \boldsymbol{R}^{-3\chi(X)+2k}$ where X denotes the underlying space of O.*

The theorem above appears in Thurston [8] with the sketch of the proof in the case that O is closed orientable.

Theorem 2. *The Teichmüller spaces of Euclidean 2-orbifolds are as follows:*

O (2-*orbifold*)	$\mathcal{T}(O)$
Torus, S^2 with 4 cone points	$\boldsymbol{R}^3$
Annulus, Möbius band, Klein bottle	$\boldsymbol{R}^2$
D^2 with 2 cone points, P^2 with 2 cone points	$\boldsymbol{R}^2$
S^2 with 3 cone points	$\boldsymbol{R}$

§2. Teichmüller spaces of geometric manifolds modelled on $H^2\times \boldsymbol{R}$, $\widetilde{SL}_2$, E^3, Nil.

Lemma 1 (Waldhausen [10]). *Let M be a Haken Seifert fibered manifold which is neither of $S^1\times S^1\times I$, $S^1\times S^1\times S^1$, the twisted I-bundle over Klein bottle, the double of the twisted I-bundle over Klein bottle, solid torus. Then the fibration of M is unique up to isotopy.*

Lemma 2 (P. Scott [7]). *Let M be a Seifert fibered manifold whose base orbifold is $S^2(p,q,r)$ where $p,q\geqq 4$. Let $f: M\to M$ be a homeomorphism homotopic to the identity. Then f is isotopic to the identity.*

Using this lemma we obtain the corollary below.

Corollary. *Let M be a manifold as in Lemma 2. Then the fibration of M is unique up to isotopy.*

Using the results of § 1, Lemma 1, and Corollary above, we can prove theorems below. The keypoint of the proof is that there is a natural fibration $\mathcal{T}(M)\to\mathcal{T}(O)$, where O denotes the base orbifold of M.

Theorem 3. *Let M be a geometric 3-manifold modelled on* $\mathrm{H}^2\times\boldsymbol{R}$, *whose base orbifold is neither $S^2(2,3,r)$ nor $S^2(3,3,r)$. Let X be the underlying space of the base orbifold. Then $T(M)\cong\boldsymbol{R}^{3-4\chi(X)+2k}$ if X is closed, and $\mathcal{T}(M)\cong\boldsymbol{R}^{2-4\chi(X)+2k}$ if X is with boundary, where k is the number of singular fibers.*

Theorem 4. *Let M be a geometric 3-manifold modelled on* $\widetilde{\mathrm{SL}}_2$, *whose base orbifold is neither $S^2(2,3,r)$ nor $S^2(3,3,r)$. Then $\mathcal{T}(M)\cong\boldsymbol{R}^{2-4\chi(X)+2k}$.*

Theorem 5. *Let M be a geometric 3-manifold modelled on* E^3, *whose base orbifold is neither $S^2(2,3,6)$ nor $S^2(3,3,3)$. We consider only metrics whose volume is equal to 1. Then the Teichmüller space is as follows:*

Base orbifold	$\mathcal{T}(M)$
Torus	$\boldsymbol{R}^5$
Klein bottle, $S^2(2,2,2,2)$	$\boldsymbol{R}^3$
Annulus	$\boldsymbol{R}^3$
$S^2(4,4,2)$	$\boldsymbol{R}$
$D^2(2,2)$, *Möbius band*	$\boldsymbol{R}^2$
$P^2(2,2)$	$\boldsymbol{R}^3$

Theorem 6. *Let M be a geometric 3-manifold modelled on* Nil, *whose base orbifold is neither $S^2(2,3,6)$ nor $S^2(3,3,3)$. Then the Teichmüller space is as follows.*

Base orbifold	$\mathcal{T}(M)$
Torus	$\boldsymbol{R}^5$
Klein bottle	$\boldsymbol{R}^4$
$S^2(2,2,2,2)$	$\boldsymbol{R}^3$
$S^2(4,4,2)$	$\boldsymbol{R}$
$P^2(2,2)$	$\boldsymbol{R}^3$

Let M be a geometric manifold modelled on one of $\mathrm{H}^2\times\boldsymbol{R}$, $\widetilde{\mathrm{SL}}_2$, E^3, Nil. Let S be the set of isotopy classes of simple closed curves and essential simple proper arcs. $\boldsymbol{R}^S$ denotes the set of all functions from S to $\boldsymbol{R}$ with weak topology. Let $\iota_*:\mathcal{T}(M)\to\boldsymbol{R}^S$ be a map such that for $m\in\mathcal{T}(M)$ s-coordinate of $\iota_*(m)$ is the logarithm of geodesic length of s on (M,m).

Theorem 7. $\iota_*:\mathcal{T}(M)\to\boldsymbol{R}^S$ *is a proper embedding.*

§ 3. Teichmüller spaces of geometric manifolds modelled on

$S^2 \times \boldsymbol{R}$. A manifold with $S^2 \times \boldsymbol{R}$-geometry is one of $S^2 \times S^1$, $P^3 \# P^3$, $D^2 \times S^1$. $S^2 \times \boldsymbol{R}$ geometry has a quite different character from other geometries. The natural fibration of $S^2 \times \boldsymbol{R}$ by lines need not descend to a Seifert fibration of a geometric manifold. So there are two ways of thinking. One way is forgetting the Seifert fibration structure of M and defining the Teichmüller space to be all $S^2 \times \boldsymbol{R}$-structure on M up to isotopy. We denote this space by $\mathcal{T}(M)$. The other way is fixing Seifert invariants of M and defining the Teichmüller space to be the subset of $\mathcal{T}(M)$ such that the natural fibration by lines on $S^2 \times \boldsymbol{R}$ descends to a Seifert fibration of M with the fixed Seifert invariants. We denote this space by $\mathcal{T}^*(M)$.

Using Laudenbach's theorems we can compute $\pi_0(\mathrm{Diff}^+(S^2 \times S^1))$, $\pi_0(\mathrm{Diff}^+(P^3 \# P^3))$, $\pi_0(\mathrm{Diff}^+(D^2 \times S^1))$. Using them we obtain the following theorems.

Theorem 8. *Let M be a geometric 3-manifold modelled on $S^2 \times \boldsymbol{R}$. Then $\mathcal{T}(M)$ is as follows:*

M	$\mathcal{T}(M)$
$S^2 \times S^1$	$S^3 \times \boldsymbol{R}$
$P^3 \# P^3$	$S^3 \times \boldsymbol{R}$
$S^1 \times D^2$	$\boldsymbol{R}^2$

Theorem 9. *Let M be a geometric 3-manifold modelled on $S^2 \times \boldsymbol{R}$. Fix Seifert invariants of M. Then $\mathcal{T}(M)$ is as follows.*

M	$\mathcal{T}^*(M)$
$S^2 \times S^1$	$\boldsymbol{R} \times \boldsymbol{Z}_2$
$P^3 \# P^3$	$\boldsymbol{R} \times \boldsymbol{Z}_2$
$D^2 \times S^1$	$\boldsymbol{R} \times \boldsymbol{Z}$

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