69. On the Ito Formula of Noncausal Type

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Let $\{B(x,w); x\geqq 0\}$ be the real Brownian motion defined on a probability space $(W,\mathcal{F},P)$ and let $\{\phi_n\}$ be an orthonormal basis in the real Hilbert space $L^2(0,1)$. Following the article [1], we say that a real random function $f(x,w)$, satisfying the condition

$$P\left[\int_0^1 f^2(x,w)dx<\infty\right]=1,$$

is integrable with respect to the basis $\{\phi_n\}$ on a measurable set $A\subset[0,1]$, if the series

$$\sum_n \int_A f(x,w)\phi_n(x)\,dx\int_0^1 \phi_n(x)dB(x)$$

converges in probability. In this case, we shall denote the sum by

$$\int_A f\,d_\phi B(x)$$

and call such integral the stochastic integral of noncausal type.

Since this integral can also apply to those random functions which are not adapted to the family of σ-fields, $\mathcal{F}_x=\sigma(B(y,w); y\leqq x)$ $(x\geqq 0)$, it is meaningful to consider the stochastic integral equation of non-causal type:

$$(1)\qquad X(x,w)-\xi(w)=\int_0^x a(y,X(y,w))dy+\int_0^x b(y,X(y,w))d_\phi B(y),$$

where $\xi(w)$ is a real random variable and $a(x,y)$, $b(x,y)$ $((x,y)\in[0,1]\times R^1)$ are some functions. As for the equation (1), Ogawa [2] has shown the existence of solutions by constructing one for a specified basis (see Theorem below). Our aim in this paper is to show that the constructed solution satisfies a formula of Ito's type in the noncausal case.

We begin by summarizing his result. Assume that the functions $a(x,y)$ and $b(x,y)$ satisfy the following two conditions:

(H, 1) The function $a(x,y)$ belongs to the class C^1 and $b(x,y)$ to the class C^2. Moreover, $b(x,y)$ is thrice continuously differentiable in y.

(H, 2) For each real number r the stochastic integral equation:

$$(2)\qquad Y(x,w)-r=\int_0^x a(y,Y(y,w))dy+\int_0^x b(y,Y(y,w))dB(y),$$

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where the term $\int b\,dB(y)$ stands for the symmetric integral, has such a solution $Y(x, w\,; r)$ almost all sample functions of which are continuous in $(x, r) \in [0,1] \times R^1$ and belong to the class C^1 in r.

Let us consider the random function $X(x, w)$, defined by $X(x, w) = Y(x, w\,; \xi(w))$ for the initial random variable $\xi(w)$. Then the following result is shown in [2],

Theorem ([2, Theorem 2]). *The random function $X(x, w)$ is a solution of the equation* (1), *provided with the trigonometric system for the basis $\{\phi_n\}$. If, in addition, the function $b(x, y)$ is of C^3-class in (x, y) and C^4-class in y, then the random function $X(x, w)$ becomes a solution of the equation* (1) *for any basis $\{\phi_n\}$.*

We now state our result.

Theorem. *Let a function $F(x)(x \in R^1)$ be in the class C^4. Then the equality:*

$$(3) \quad F(X(x, w)) - F(\xi(w)) = \int_0^x F'(X(y, w))\{a(y, X(y, w))dy + b(y, X(y, w))d_\phi B(y)\}$$

holds, provided with the trigonometric system for the basis $\{\phi_n\}$. Moreover, if the function $b(x, y)$ is of C^3-class in (x, y) and C^4-class in y, and if the function $F(x)$ is in the class C^5, then the above equality holds for any basis $\{\phi_n\}$.

Proof. Applying the usual Ito formula to the solution $Y(x, w\,; r)$ of the equation (2), we find

$$F(Y(x, w\,; r)) - F(r) = \int_0^x F'(Y(y, w\,; r))\{a(y, Y(y, w\,; r))dy + b(y, Y(y, w\,; r))dB(y)\}.$$

We set

$$F_n^\phi(x, w\,; r) = F(r) + \int_0^x F'(Y(y, w\,; r)) \times \{a(y, Y(y, w\,; r))dy + b(y, Y(y, w\,; r))dB_\phi^n(y)\},$$

where

$$B_\phi^n(x, w) = \sum_{k=0}^n (\phi_k, \dot{B}) \int_0^x \phi_k(y)dy,$$

and notice that

$$\frac{\partial}{\partial r}\{F(Y(x, w\,; r)) - F_n^\phi(x, w\,; r)\} = \int_0^x \{F''(Y(y, w\,; r))b(y, Y(y, w\,; r)) + F'(Y(y, w\,; r))b'(y, Y(y, w\,; r))\}Y'(y, w\,; r)(dB(y) - dB_\phi^n(y)),$$

where

$$Y'(y, w\,; r) = \frac{\partial}{\partial r} Y(y, w\,; r) \quad \text{and} \quad b'(x, y) = \frac{\partial}{\partial y} b(x, y).$$

Then, using the same argument as in the proof of [2, Proposition 2 and Theorem 2], we can derive the equality (3), but we omit the details here.

Finally we shall remark on the uniqueness of solutions of the equation:

$$X(x,w)-\xi(w)=\int_0^x a(y,X(y,w))dy+\int_0^x b(X(y,w))d_\phi B(y). \tag{4}$$

In addition to the assumptions (H, 1) and (H, 2), we suppose that the function $b(x)$ is positive. Then the solution $X(x,w)$ of the equation (4), which satisfies the Ito formula (3) of noncausal type, is unique. Indeed, setting

$$F(x)=\int_0^x (1/b(y))dy$$

in the formula (3), we obtain the equation:

$$F(X(x,w))-F(\xi(w))=\int_0^x \{a(y,X(y,w))/b(X(y,w))\}dy+B(x,w),$$

which admits only one solution.

References

[1] S. Ogawa: Sur le produit direct du bruit blanc par lui-même. C. R. Acad. Sc. Paris, **288**, Série A, 359–362 (1979).

[2] ——: Sur la question d'existence de solutions d'une équation différentielle stochastique du type noncausal. J. Math. Kyoto Univ., vol. 24, no. 4 (1984).