66. *Continuity of the Inverse of a Certain Integral Operator*

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§1. Let $L=\bigcup L_j$ be a union of a finite number of simple, smooth and bounded open arcs in $\mathbf{R}^2$, where any two of L_j have neither an interior point nor an end point in common. Denote points in $\mathbf{R}^2$ by x, y, etc., and the distance between x and y by $|x-y|$. Let $\partial L=\{x^*\}$ be the set of end points x^* of L, and set $\bar{L}=L\cup\partial L$. Suppose $C=C(\bar{L})$, $C^\infty=C^\infty(\bar{L})=\mathcal{E}(\bar{L})$, $C_0^\infty=C_0^\infty(\bar{L})=\mathcal{D}(\bar{L})$, etc., represent the function spaces on $\bar{L}$ in the usual sense.

Assume $\psi(x, y)=(1/4i)H_0^{(2)}(k|x-y|)$, where $H_0^{(2)}$ is the zero-th order Hankel function of the second kind, and k is a constant such as $\operatorname{Im} k \leqq 0$. ψ is a fundamental solution of the Helmholtz equation.

We shall define an integral operator Ψ by

(1) $$\Psi\tau\equiv\int_L \psi(x, y)\tau(y)ds_y$$

and denote the inverse of Ψ by Ψ^{-1}. The purpose of this work is to study about the continuity of Ψ^{-1}.

Since $\psi(x, y)$ has only a log singularity at $x=y$, Ψ maps $C(\bar{L})$ into $C(\bar{L})$. Furthermore, as was proved in the previous paper [1], $\Psi\tau=0$ is equivalent to $\tau=0$. However, as is implied by the Riemann-Lebesgue theorem, Ψ^{-1} is not necessarily continuous. For example, for $x\neq a$, we have

$$\int_0^a \psi(x, y)\cos my\,\mathrm{d}y=\left(\frac{1}{m}\right)\psi(x, a)\sin ma-\left(\frac{1}{m}\right)\int_0^a \frac{\partial\psi(x, y)}{\partial y}\sin my\,\mathrm{d}y.$$

The right hand side exists in the sense of Cauchy's principal value of integral, which tends to zero as $m\to\infty$. However, $\cos mx$ does not tend to zero in $C([0, a])$. In contrast with this, we shall show that Ψ^{-1} is continuous if Ψ is considered to map $\mathcal{D}\to\mathcal{E}$.

§2. **Definition 1.** Set $\psi(x, y)=\psi_0(x, y)=\psi^{[0]}(x, y)$, where ψ is the one defined above, and set

$$\psi_m(x, y)=\int^{s_y}\psi_{m-1}(x, z)ds_z,$$

and

$$\psi^{[m]}(x, y)=\frac{\partial}{\partial s_x}\int^{s_y}\psi^{[m-1]}(x, z)ds_z, \qquad (m=1, 2, \cdots),$$

where $\int^{s_y}\{\quad\}ds_z$ is the integration with respect to the arc element ds_z

of a point z till a point $y \in L$, while $\partial/\partial s_x$ is the tangential differentiation at x.

Lemma 1.

(2) $\psi^{[m]}(x, y) = c_m \cdot \log|x-y| + f_m(x, y)|x-y| \cdot \log|x-y| + g_m(x, y)$,

where $f_m(x, y),\ g_m(x, y) \in C^2(L \times L)$, $c_m = (-1)^m/2\pi$, *and* $m = 1, 2, \cdots$.

Proof. For $m=0$, (2) is obtained from the expansion formula for $H_0^{(2)}$. For $m=m$, (2) is shown to hold by mathematical induction.

Lemma 2.

$$\frac{\partial^m \psi_m(x, y)}{\partial s_x^m} = \psi^{[m]}(x, y). \tag{3}$$

Proof. The proof is straightforward if one note that $\psi^{[m]}(x, y)$ has only a log singularity.

Definition 2. For $\sigma \in C_0^\infty$, set $\hat{\sigma} = \Psi\sigma$. As usual, m-th order derivatives are described as

$$\sigma^{(m)}(x) = \frac{d^m \sigma(x)}{ds_x^m} \quad \text{and} \quad \hat{\sigma}^{(m)}(x) = \frac{d^m \hat{\sigma}(x)}{ds_x^m}.$$

Note that $\widehat{\sigma^{(m)}}(x) = \Psi\sigma^{(m)}(x)$ is different from $\hat{\sigma}^{(m)}(x)$.

Definition 3. $\hat{\Sigma} = \{\hat{\sigma}\,;\ \hat{\sigma} = \Psi\sigma,\ \sigma \in C_0^\infty\}$.

Theorem 1. *For* $\forall\sigma \in C_0^\infty$, *we have*

$$\hat{\sigma}^{(m)}(x) = (-1)^m \cdot \int_L \psi^{[m]}(x, y)\sigma^{(m)}(y)ds_y \in C(\bar{L}). \tag{4}$$

Proof. By integrating by parts,

$$\hat{\sigma}(x) = \int_L \psi(x, y)\sigma(y)\,ds_y = (-1)\cdot\int_L \psi_1(x, y)\sigma^{(1)}(y)ds_y$$
$$= \cdots = (-1)^m \cdot \int_L \psi_m(x, y)\sigma^{(m)}(y)ds_y.$$

Consequently, by Lemma 2, we have (4).

Corollary 1. Ψ *maps* C_0^∞ *into* C^∞. *That is,* $\hat{\Sigma} \subset C^\infty$.

Note. If $\tau \in C$, then $\hat{\tau} = \Psi\tau \in C$. However, $\hat{\tau}$ does not necessarily belong to C^2.

Suppose L' is a pertinent union of open arcs such that $C = \bar{L} \cup L'$ is a closed contour, or a union of closed contours.

Theorem 2. *For* $\forall\tau \in C(\bar{L})$ *and* $\forall\phi \in C_0^2(\bar{L})$, *we have the following identity*;

$$\int_L \phi(x)ds_x\left[\tau(x) + \int_L K(x, y)\tau(y)ds_y\right] = \int_L \hat{\tau}(x)ds_x \int_L \lambda(x, y)\phi(y)ds_y. \tag{5}$$

Here we have set $\hat{\tau} = \Psi\tau$, *and*

$$K(x, y) = 4\left[\int_{L'} \frac{\partial^2\psi(x, z)}{\partial n(x)\partial n(z)}\psi(y, z)ds_z - \int_C \frac{\partial\psi(x, z)}{\partial n(x)}\frac{\partial\psi(y, z)}{\partial n(z)}ds_z\right]. \tag{6}$$

$$\lambda(x, y) = -4\frac{\partial^2\psi(x, y)}{\partial n(x)\partial n(y)}.$$

where $\partial/\partial n(x)$ *denotes the differentiation along the normal* n *of* L *at* x.

Note. (5) holds as well if τ is piecewise continuous on L.

Proof. Let

$$v(x)=\int_L \psi(x, y)\tau(y)ds_y, \qquad w(x)=\int_L \frac{\partial\psi(x, y)}{\partial n(y)}\phi(y)ds_y.$$

Then, (5) is derived by the Green's second identity applied to v and w in the domain exterior to $C=\bar{L}\cup L'$.

As is well known, for $\forall\hat{\tau}\in C(\bar{L})$, there exists a sequence $\hat{\tau}_m\in C^\infty(\bar{L})$ such that $\|\hat{\tau}-\hat{\tau}_m\|=\sup_L|\hat{\tau}(x)-\hat{\tau}_m(x)|\to 0$ when $m\to\infty$.

Theorem 3. *Let τ be piecewise continuous on L, and set $\hat{\tau}=\Psi\tau$. Then, the following identity holds in the sense of a distribution defined on $\mathcal{D}=C_0^\infty(\bar{L})$,*

$$(7)\qquad \tau(x)+\int_L K(x, y)\tau(y)\,ds_y=\lim_{m\to\infty}\int_L \lambda(x, y)\hat{\tau}_m(y)ds_y.$$

Proof. The right hand member of (5) is rewritten as

$$\lim\int_L \phi(x)ds_x\int_L \lambda(x, y)\hat{\tau}_m(y)ds_y,$$

because $\hat{\tau}_m$ tends uniformly to $\hat{\tau}$, $\lambda(x, y)=\lambda(y, x)$ and the order of integrations is interchangeable for $\hat{\tau}_m\in C^\infty$. Consequently, by virtue of the completeness of the space $\mathcal{D}'$, we have (7).

Corollary. *For $\forall\sigma\in C_0^\infty$, the following identity holds in the sense of distribution,*

$$(8)\qquad \sigma(x)+\int_L K(x, y)\sigma(y)ds_y=\int_L \lambda(x, y)\hat{\sigma}(y)ds_y,$$

where $\hat{\sigma}=\Psi\sigma\in C^\infty$.

Note. $K(x, y)$ and $\lambda(x, y)$ are not necessarily bounded at an end point x^* of L.

Definition 4. For $\forall\rho>0$, set $L_\rho=\{x\,;\,x\in L,\ |x-x^*|\geqq\rho,\ x^*\in\partial L\}$, and $L_\rho^c=L-L_\rho$.

Lemma 3. *For $\forall\hat{\tau}\in C(\bar{L})$, let $\hat{\tau}_m\in C^\infty(\bar{L})$ be the sequence mentioned above, and set*

$$\frac{\partial w_m(x)}{\partial n(x)}=\int_L \frac{\partial^2\psi(x, y)}{\partial n(x)\partial n(y)}\hat{\tau}_m(y)ds_y.$$

If $\lim_{m\to\infty}\partial w_m/\partial n=0$ holds at $\forall x\in L_\rho$ for sufficiently small ρ, then, $\hat{\tau}(x)=0$ holds for $\forall x\in\bar{L}$.

Proof. This is proved by the study of the behavior of $\partial w_m/\partial n$ near an end point x^*. However, the detailed proof of this important lemma is too long to describe here.

Note. Though the kernel $K(x, y)$ defined by (6) is not bounded at end points x^*, it is continuous with respect to x and y if $x, y\in L_\rho$. That is, the operator

$$K\phi\equiv\int_L K(x, y)\phi(y)ds_y$$

is completely continuous when it maps $C(L_\rho)\to C(L_\rho)$.

Theorem 4. *Set*

$$(I+K)\tau \equiv \tau(x)+\int_L K(x, y)\tau(y)\,ds_y, \qquad x \in L_\rho,$$

then, the inverse $(I+K)^{-1}$; $C(L_\rho)\to C(L_\rho)$ *exists and is continuous.*

Proof. For $\tau \in C(L_\rho)$, we have (7). If $(I+K)\tau=0$, then

$$\lim_{m\to\infty} \frac{\partial w_m}{\partial n}=0$$

follows from the right hand side of (7). Consequently, by Lemma 3, we have $\hat{\tau}(x)=0$. While, as was proved in [1], $\hat{\tau}=0$ is equivalent to $\tau=0$.

§ 3. With help of these results obtained above, we can prove the following theorem.

Definition 5. $\sigma\to 0$ in $\mathcal{D}$ means $\|\sigma^{(m)}\|=\sup|\sigma^{(m)}|\to 0$ for $m=0, 1, 2, \cdots$. Similarly, $\hat{\sigma}\to 0$ in $\mathcal{E}$ means $\|\hat{\sigma}^{(m)}\|\to 0$ for $m=0, 1, 2, \cdots$.

Theorem 5. $\hat{\sigma}\to 0$ *in* $\mathcal{E}$ $\rightleftarrows$ $\sigma\to 0$ *in* $\mathcal{D}$.

Proof. By virtue of Theorem 1, it is easy to see that $\sigma\to 0$ in $\mathcal{D}\Rightarrow\hat{\sigma}\to 0$ in $\mathcal{E}$. The converse is also true. A brief proof is as follows; Let $\rho>0$ be an arbitrarily fixed constant, and set $\mathcal{D}_\rho=\{\sigma; \sigma \in \mathcal{D}, \operatorname{supp}\sigma\subset L_\rho\}$. For $\sigma \in \mathcal{D}_\rho$, (8) holds, and we have, by Theorem 4, $\sigma=(I+K)^{-1}\hat{\sigma}$. Consequently, $\|\sigma\|\to 0$ follows from $\|\hat{\sigma}\|\to 0$. Assume that $\|\hat{\sigma}^{(p)}\|\to 0$ implies $\|\sigma^{(p)}\|\to 0$ for $p=0, 1, \cdots, m-1$. Then, with help of Theorem 1, it is proved that, when $\|\hat{\sigma}^{(m)}\|\to 0$, we have

$$\left\|\int_L \lambda(x, y)\widehat{\sigma^{(m)}(y)}\,ds_y\right\|\to 0.$$

On the other hand, from (8), we have

$$\sigma^{(m)}=(I+K)^{-1}\cdot\int_L \lambda(x, y)\widehat{\sigma^{(m)}(y)}\,ds_y.$$

Consequently, $\|\sigma^{(m)}\|\to 0$ follows from $\|\hat{\sigma}^{(p)}\|\to 0$, $p=0, 1, \cdots, m$.

Reference

[1] Hayashi, Y.: The Dirichlet problem for the two-dimensional Helmholtz equation for an open boundary. J. Math. Anal. Appl., 44, 489–530 (1973).