65. *On the Ergodicity of Solutions of Nonlinear Evolution Equations with Periodic Forcings*

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§ 1. Introduction and statement of the result. In this note, we consider the asymptotic behavior of the solution of the initial value problem

$$(1)\qquad \frac{du(t)}{dt}+Au(t)\ni f(t),\quad 0\leq t<\infty,\quad u(0)=u_0,$$

where A is an m-accretive operator [2] on a real Banach space E, $u_0\in\overline{D(A)}$ (domain of A) and $f:[0,+\infty)\to E$ is a periodic function. The ergodicity of the solution of (1) in case that $f=0$ has been studied by several authors (cf. [1], [6]). In [5], Miyadera and Kobayashi established an ergodic theorem for the solution of (1) in case that $f\in L^1(0,\infty;E)$ ($L^1(0,\infty;E)$ denotes the space of all E-valued functions $u:[0,\infty)\to E$ such that $\int_0^\infty\|u(t)\|\,dt<+\infty$). Also, Kobayashi [4] obtained mean convergence theorems for the solution of (1). We give the following nonlinear ergodic theorem for the solution of (1) by using the nonlinear ergodic theorem due to Reich [6].

Theorem. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and A be an m-accretive operator on E. Let $u_0\in\overline{D(A)}$ and $f\in L^1_{\rm loc}(0,\infty;E)$ be a periodic function. Suppose that the weak solution u of* (1) *is bounded in E* (i.e., $\sup_t\|u(t)\|<\infty$). *Then $(1/t)\int_0^t u(s)\,ds$ converges weakly to a point in E, as $t\to\infty$.*

§ 2. Proof. Let $T>0$ be the period of f. For each $s\in[0,T]$, we define an operator $G(s):\overline{D(A)}\to\overline{D(A)}$ by $G(s)u=v_s(T)$, where $v_s:[0,T]\to E$ is the weak solution [2] of the initial value problem

$$(2)\qquad \frac{dv_s(t)}{dt}+Av_s(t)\ni f(s+t),\quad 0<t\leq T,\quad v(0)=u.$$

Then it is easy to see that $G(s)$ is nonexpansive for each $s\in[0,T]$. In fact, from Theorem 2.1 in Chapter III of [2], we find that $\|G(s)u-G(s)v\|\leq\|u-v\|$ for all $u,v\in\overline{D(A)}$ and $s\in[0,T]$. On the other hand, we have from the periodicity of the function f that $G(s)^k u(s)=u(kT+s)$, for $s\in[0,T]$ and $k=0,1,2,\cdots$. Thus from the hypothesis, we have that for each $s\in[0,T]$, $\{G(s)^k u(s)\}_{k=0}^\infty$ is bounded in E. Hence by using the mean ergodic theorem in [6], we obtain that for each

$s \in [0, T]$, $(1/n)\sum_{k=0}^{n-1} G(s)^k u(s)$ converges weakly to a point $w(s) \in E$, as $n \to \infty$. In other words, $(1/n)\sum_{k=0}^{n-1} u(kT+s)$ converges weakly to $w(s)$, as $n \to \infty$. For each $n \geq 1$, we define a function $h_n : [0, T] \to E$ by $h_n(s) = (1/n)\sum_{k=0}^{n-1} u(kT+s)$ for $s \in [0, T]$. Then $h_n : [0, T] \to E$ is a continuous function and $\|h_n(s)\| \leq \sup_t \|u(t)\|$ for all $s \in [0, T]$ and $n \geq 1$. Since $h_n(s) \to w(s)$ weakly as $n \to \infty$, for each $s \in [0, T]$, we deduce by Lebesgue's bounded convergence theorem that for each $z \in E'$,

$$\lim_{n\to\infty} \int_0^T (z, h_n(s))\, ds = \int_0^T (z, w(s))\, ds.$$

In other words,

$$\lim_{n\to\infty} \left((1/n)\int_0^T \sum_{k=0}^{n-1} u(kT+s)\, ds,\ z\right) = \left(z, \int_0^T w(s)\, ds\right).$$

Since

$$(1/n)\int_0^T \sum_{k=0}^{n-1} u(kT+s)\, ds = (1/n)\int_0^{nT} u(s)\, ds,$$

we find that

$$(1/nT)\int_0^{nT} u(s)\, ds$$

converges weakly to a point

$$w = (1/T)\int_0^T w(s)\, ds,$$

as $n \to \infty$. Then it follows that

$$(1/t)\int_0^t u(s)\, ds$$

converges weakly to a point $w \in E$, as $t \to \infty$.

Remark. The assumption that E has a Fréchet differentiable norm can be replaced by the assumption that E satisfies Optial's condition [3]. In [4], Kobayashi showed $w(s)$ is periodic, i.e., $w(0) = w(T)$.

References

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