37. On Marot Rings

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§1. **Introduction.** Throughout the paper, a ring means a commutative ring with identity. A non-zerodivisor of a ring is said to be regular, and an ideal containing regular elements is said to be regular. A ring R is said to be a *Marot ring* (cf. [3]), if each regular ideal of R is generated by regular elements. The main purpose of this paper is to solve the following question on Marot rings posed by Portelli-Spangher [6]. : Let $\mathfrak{a}$ be an ideal of a ring R. We denote the set of regular elements contained in $\mathfrak{a}$ by $\mathrm{Reg}(\mathfrak{a})$. We say that a ring R has *property* (FU), if $\mathrm{Reg}(\mathfrak{a})\subset\bigcup_{i=1}^{n}\mathfrak{a}_i$ implies $\mathfrak{a}\subset\bigcup_{i=1}^{n}\mathfrak{a}_i$ for each family of a finite number of regular ideals $\mathfrak{a},\mathfrak{a}_1,\mathfrak{a}_2,\cdots,\mathfrak{a}_n$. If R has property (FU), then R is a Marot ring. The question is: *Does a Marot ring have property* (FU)?

§2. **Answer to the question.** Let us begin by some lemmas.

Lemma 1. *Let R be a ring.*

(1) *R is a Marot ring if and only if an ideal (r, s) is generated by regular elements for each regular element r of R and for each element $s\in R$.*

(2) *R has property (FU) if and only if* $\mathrm{Reg}((r,s))\subset\bigcup_{i=1}^{n}\mathfrak{a}_i$ *implies $(r,s)\subset\bigcup_{i=1}^{n}\mathfrak{a}_i$ for each pair of elements r, s of R with r regular and for each family of a finite number of regular ideals $\mathfrak{a}_1,\mathfrak{a}_2,\cdots,\mathfrak{a}_n$.*

Let A be a ring, and let M be an A-module. We construct a semidirect product R by the principle of idealization ([5, Chap. 1, n°1]). That is, $R=A\oplus M$ and for elements $f+x$ and $g+y$ of R we set $(f+x)(g+y)=fg+(fy+gx)$, where $f,g\in A$ and $x,y\in M$.

Lemma 2. *Let $f+x$ be an element of R. Then $f+x$ is a regular element of R if and only if f is a regular element of R.*

Let p be a prime number, and let k be a finite field of characteristic p. We denote by A the subring $k[X^p, X^{p+1}, X^{p+2},\cdots]$ of the polynomial ring $k[X]$. Let $\{F_0, F_1,\cdots,F_n, G_1, G_2,\cdots\}$ be a set of irreducible polynomials of $k[X]$ such that (1) $F_0=X$ and $F_1=1+X$, (2) $\deg(F_i)<2p$ for each i, (3) $\deg(G_j)\geq 2p$ for each j, (4) any two elements of the set are not associated and (5) each irreducible polynomial of $k[X]$ is associated with some element of the set. We denote $k[X]/(G_j)$ by K_j. K_j is naturally an A-module. We construct a direct sum M of A-modules $K_1, K_2, K_3,\cdots$, and construct a semidirect product $R=A$

$\oplus M$. R will keep this meaning in the next four lemmas.

Lemma 3. *The set of regular elements of R is $\{a+x\,;\,0\neq a\in k, x\in M\}\cup\{aX^eF_1^{e_1}F_2^{e_2}\cdots F_n^{e_n}+x\,;\,0\neq a\in k,\ e\geq p,\ e_i\geq 0,\ x\in M\}$.*

Lemma 4. *Let F, F', G and G' be elements of $k[X]$ such that $FF'\in A$. If FG and $F'G'$ are regular elements of R, then FF' is a regular element of R.*

Lemma 5. *If $r\in R$ is regular, we have $rM=M$.*

Lemma 6. *R is a Marot ring.*

Proof. Let $\mathfrak{a}$ be a regular ideal of R. By Lemma 1(1), we may assume that $\mathfrak{a}$ is generated by two elements r and s of R with r regular. We set $r=f+x$, $s=g+y$ for $f, g\in A$ and $x, y\in M$. By Lemma 5, we have $\mathfrak{a}=(f, g)$. Let D be a greatest common divisor of f and g in $k[X]$. We have $D=fF+gG$ for some $F, G\in k[X]$. Therefore DX^p belongs to $\mathfrak{a}$. It follows that $\mathfrak{a}\ni DX^{2p}, DX^{2p+1}, DX^{2p+2}, \cdots$. We have $g=DG'$ for some $G'\in k[X]$. Set $G'=a_lX^l+a_{l+1}X^{l+1}+\cdots+a_mX^m$ with $a_l\neq 0$ and $a_m\neq 0$. If $l\geq 2p$, $\mathfrak{a}$ is generated by regular elements $f, DX^l, DX^{l+1}, \cdots, DX^m$. Suppose that $l<2p$. Then we have $\mathfrak{a}=(f, DX^p, DG'')$ for some $G''\in k[X]$ degree of which is less than $2p$. By Lemmas 3 and 4, DG'' is a regular element of R. Therefore $\mathfrak{a}$ is generated by regular elements.

We set $\{X^eF_1^{e_1}F_2^{e_2}\cdots F_n^{e_n}\,;\,p\leq e<2p,\ 0\leq e_i<p\}=\{f_1, f_2, \cdots, f_h\}$, where $h=p^{n+1}$. And we set $\mathfrak{a}_0=(X^p, X^{p+1}, \cdots, X^{2p-1})$ and $\mathfrak{a}_i=(f_i)$ for $1\leq i\leq h$.

Lemma 7. *We have* $\operatorname{Reg}(\mathfrak{a}_0)\subset\bigcup_{i=1}^h\mathfrak{a}_i$.

Lemma 8. *Let k be a prime field of characteristic 2. Then we have $X^2+X^3+\cdots+X^l\in\mathfrak{a}_0-\bigcup_{i=1}^h\mathfrak{a}_i$ for each even natural number $l>5$.*

If k is not a prime field of characteristic 2, there exist nonzero elements a and b of k such that $a+b\neq 0$.

Lemma 9. *Set $f=aX^pF_1F_2^pF_3^p\cdots F_n^p+bX^p$. Then $f\in\mathfrak{a}_0-\bigcup_{i=1}^h\mathfrak{a}_i$.*

Proof. If f belongs to some $\mathfrak{a}_i$, we have $f=X^eF_1^{e_1}\cdots F_n^{e_n}g$ for $p\leq e<2p$, $0\leq e_i<p$ and $g\in A$. Since $b\neq 0$, we have $aF_1F_2^p\cdots F_n^p+b=X^{e-p}g$. Since $a+b\neq 0$, we have $aF_1F_2^p\cdots F_n^p+b=g$. Since $a\neq 0$ and $F_i^p\in A$ for each i, it follows that $F_1\in A$; which is a contradiction.

Lemmas 6–9 imply the following answer to Question:

Theorem 10. *There exist Marot rings which do not have property (FU).*

§3. **Some other results.** A ring R is said to be an *additively regular ring*, if for each pair of elements $r, s\in R$ with r regular there exists $r'\in R$ such that $r'r+s$ is a regular element ([2]). An additively regular ring has property (*FU*) ([6, Proposition 8]).

Proposition 11. *There exists a ring R with property (FU) which is not an additively regular ring.*

Proof. We construct a direct sum M of Z-modules $Z, Z/(3), Z/(5)$,

$Z/(7), \cdots$, and construct a semidirect product $R=Z\oplus M$. Let e and n be natural numbers with n odd such that $2<n<2^e-1$. Then 2^e is a regular element of R, and $r2^e+n$ is not a regular element for each $r\in R$. Therefore R is not an additively regular ring. The following assertion implies that R has property (FU).

Proposition 12. *Let A be a principal ideal domain, and M an A-module. Then a semidirect product $A\oplus M$ has property* (FU).

Proof. Set $R=A\oplus M$. Let $\mathfrak{a}, \mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_n$ be regular ideals of R such that $\operatorname{Reg}(\mathfrak{a})\subset\bigcup_{i=1}^n \mathfrak{a}_i$. By Lemma 1(2), we may assume that $\mathfrak{a}$ is generated by two elements r and s with r regular. We set $r=a+x$ and $s=b+y$ for $a, b\in A$ and $x, y\in M$. We have $(a, b)A=dA$ for some $d\in A$. There exists $x'\in M$ such that $d+x'\in\mathfrak{a}$. Therefore we may assume that $b=0$. Each element r' of $\mathfrak{a}$ is of the form: $aa'+(ax'+a'x+b'y)$ for $a', b'\in A$ and $x'\in M$. If a' is either zero or a unit of A, it is not difficult to see that $r'\in\bigcup_{i=1}^n \mathfrak{a}_i$. If a' is neither zero nor a unit, we can write $a'=a_1a_2$, where each irreducible factor of a_1 (resp. a_2) in A is (resp. is not) a regular element of R. Since $(a_1, a_2)A=A$, we have $r'=[aa_1+(ab_1x'+a_1x+b_2y)]\,[a_2+(b_3x'+b_4y)]$ for some $b_1, b_2, b_3, b_4\in A$. Since $aa_1+(ab_1x'+a_1x+b_2y)$ is a regular element contained in $\mathfrak{a}$, r' belongs to some $\mathfrak{a}_i$. Therefore R has property (FU).

If we replace A by a Bezout domain (that is, an integral domain each finitely generated ideal of which is a principal ideal) in Proposition 12, we have:

Remark 13. Let A be a Bezout domain, and M an A-module. Then a semidirect product $R=A\oplus M$ is a Marot ring.

Proof. Let $\mathfrak{a}$ be a regular ideal of R. We may assume that $\mathfrak{a}$ is generated by two elements $r, s\in R$ with r regular. We set $r=a+x$, $s=b+y$ and $(a, b)A=dA$. There exists $x'\in M$ such that $d+x'\in\operatorname{Reg}(\mathfrak{a})$. Each element r' of $\mathfrak{a}$ is of the form: $dd'+y'$. We have $r'=(d+x')(d'-1)+[d+(1-d')x'+y']$. Therefore $\mathfrak{a}$ is generated by regular elements.

We say that a ring R has *property* (U), if each regular ideal of R is a (set-theoretical) union of regular principal ideals. A ring R with property (U) has property (FU).

Proposition 14. (1) *There exists a ring R with property (FU) which is not an additively regular ring and has not property* (U).

(2) *There exists an additively regular ring which does not have property* (U).

(3) *There exists a ring with property (U) which is not an additively regular ring.*

Proof. (1) We consider the ring R of Proposition 11. We denote the element 1 of Z contained in M by x_0, and set $\mathfrak{a}=(4+x_0, 2x_0)$.

Then $\mathfrak{a}$ is not a union of regular principal ideals of R. Therefore R does not have property (U). (2) We set $A=(Z/(4))[X]$ and set $P=(2, X)A$. Then A_P is an additively regular ring which does not have property (U). A_P is a Noetherian local ring. (3) We construct a direct sum M of Z-modules $Z/(3), Z/(5), Z/(7), \cdots$, and construct a semidirect product $R=Z\oplus M$. Then R is a desired ring.

We note finally that we have studied Marot rings also in [4] and solved another question posed in [6]: Generalize Theorem 2 in § 1.3 in [1] to a ring with zerodivisors. This is done in the proof of Theorem (7.1) in [4].

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