28. On n-Unitary Subsemigroups of Semigroups

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Let S denote a semigroup and H a subset of S. Using notation $H\cdots a=\{(x,y)\in S\times S: xay\in H\}$ for all elements a in S, it can be easily verified that $P_H=\{(a,b)\in S\times S: H\cdots a=H\cdots b\}$ is a congruence on S. P_H is called the principal congruence on S determined by H ([1]).

In [2] it is shown that if H is a reflexive unitary subsemigroup of a semigroup S, then S/P_H is either a group or a group with zero. Conversely, if P is a congruence on a semigroup S such that S/P is a group or a group with zero, with identity H, then H is a reflexive unitary subsemigroup of S and $P_H=P$ (Theorem 1.1 of [2]).

In [2] we also proved that if H and N are unitary subsemigroups of a semigroup S such that H is reflexive in S, then $H\cap N$ is either empty or a reflexive unitary subsemigroup of N and $\langle H,N\rangle/P_H$ is isomorphic with $N/P_{H\cap N}$. If N is also reflexive in S, then N/P_H is a normal subgroup of S/P_H and $(S/P_H)/(N/P_H)$ is isomorphic with S/P_N (Theorem 1.5 of [2]).

The mentioned results suggest that the simple reflexive unitary subsemigroups of semigroups can play a similar role to the normal subgroups of groups. But, as the following example shows, it is necessary to make the conditions stronger. Let $S_1(\ominus)$ and $S_2(+)$ be (completely) simple semigroups with $S_1\cap S_2=\phi$. Let 0 denote a symbol, $0\notin S_1$ and $0\notin S_2$. On the set $S=S_1\cup S_2\cup\{0\}$, we define an operation. For every $t,s\in S$, let

$$ts=\begin{cases} t\ominus s & \text{if } t,s\in S_1,\\ t+s & \text{if } t,s\in S_2,\\ 0 & \text{in other cases.}\end{cases}$$

It can be easily verified that S_1 and S_2 are (completely) simple unitary subsemigroups of S and $S_1S_2=S_2S_1=0$ is not unitary in S. We note that $\langle S_1,S_2\rangle=S\neq S_1S_2$.

Denote $U(S)$ the set of those unitary subsemigroups of the semigroup S all of whose unitary subsemigroups are simple. As in [2], we say that a semigroup H of S is an *n-unitary subsemigroup* of S if

(a) $H\in U(S)$ and H is reflexive in S,

(b) $V\in U(S)$ implies $\langle H,V\rangle=HV\in U(S)$.

In [2] it is shown that

Lemma 1. *If $H\subseteq N$ are n-unitary subsemigroups of a semigroup,* *then H is n-unitary in N, too* (Theorem 2.2 of [2]).

Lemma 2. *If H and N are n-unitary subsemigroups of a semigroup S, then the semigroup $\langle H, N\rangle$ of S generated by H and N is also n-unitary in S* (Theorem 2.4 of [2]).

Lemma 3. *If H and N are n-unitary subsemigroups of a semigroup S, then $H\cap N$ is an n-unitary subsemigroup of S and $\langle H, N\rangle/P_H$ is isomorphic with $N/P_{H\cap N}$* (Theorem 2.3 of [2]).

In this paper we formulate an isomorphism theorem concerning the n-unitary subsemigroups of semigroups. First we show that there is an isomorphism between $\langle H, N\rangle/P_H$ and $N/P_{H\cap N}$ in that case when N is only a subsemigroup and N is a reflexive unitary subsemigroup of a semigroup, except for $H\cap N=\phi$. Then, generalizing Lemma 3, we prove that if A is an n-unitary subsemigroup of a semigroup S and $M\in U(S)$, then $A\cap M$ is an n-unitary subsemigroup of M (and $\langle A, M\rangle/P_A$ is isomorphic with $M/P_{A\cap M}$). These results and Theorem 1.5 of [2] are generalizations of isomorphism theorems concerning the normal subgroups of the groups.

Notations. If H is a subset of a subsemigroup N of a semigroup S, then the principal congruence on N determined by H will be denoted by $P_H(N)$. If $N=S$, then we shall use P_H instead of $P_H(S)$. Moreover, we use N/P_H instead of $N/P_H(N)$. For notations and notions not defined here, we refer to [1].

The following lemma shows that we need not distinguish $P_H(N)$ from P_H/N if N is a unitary and H is a reflexive unitary subsemigroup of a semigroup S with $H\subseteq N$. Here P_H/N denotes the restriction of P_H to N.

Lemma 4. *If $H\subseteq N$ are unitary subsemigroups of a semigroup S and H is reflexive in S, then $P_H(N)=P_H/N$.*

Proof. Evidently, $P_H/N\subseteq P_H(N)$. Let a and b be arbitrary elements of N with $(a, b)\in P_H(N)$. We prove that $(a, b)\in P_H/N$. Assume, in an indirect way, that $(a, b)\notin P_H/N$.

Then there are elements x and y in S such that, for example, $xay\in H$ and $xby\notin H$. In case $xay\notin H$, $xby\in H$, the proof will be similar. Since H is reflexive in S, it follows that $yxa\in H$ and $yxb\notin H$. As $a\in N$ and $yxa\in H\subseteq N$, we have $yx\in N$, because N is unitary in S. Thus, for an arbitrary element h in H, it follows that $yxah\in H$ and $yxbh\notin H$. Since yx, $h\in N$, we have $(a, b)\notin P_H(N)$. But it is a contradiction. Consequently $(a, b)\in P_H/N$, that is $P_H(N)=P_H/N$.

Lemma 5. *If H and N are subsemigroups of a semigroup S such that N is reflexive and unitary in S and $H\cap N\neq\phi$, then $\langle H, N\rangle/P_N$ is isomorphic with $H/P_{H\cap N}$.*

Proof. Let P_N denote the principal congruence on $\langle H, N\rangle$ determined by N. Let $(a, b) \in P_N$ for $a, b \in H$. Then, for every $x, y \in H$, $xay \in H \cap N$ if and only if $xby \in H \cap N$. So $(a, b) \in P_{H \cap N}(H)$, that is $P_N/H \subseteq P_{H \cap N}(H)$.

We show that $P_{H \cap N}(H) \subseteq P_N/H$. Let $a, b \in H$ with $(a, b) \in P_{H \cap N}(H)$. Assume, in an indirect way, that $(a, b) \notin P_N/H$. Then there are elements x, y in $\langle H, N \rangle$ such that, for example, $xay \in N$ and $xby \notin N$. Since N is the identity element of $\langle H, N \rangle / P_N$, there are elements u and v in H with $(x, u) \in P_N$ and $(y, v) \in P_N$. Using $(xay, uav) \in P_N$ and $(xby, ubv) \in P_N$ several times, we have $uav \in N$ and $ubv \notin N$. Since $u, v \in H$, this contradicts $(a, b) \in P_{H \cap N}(H)$. Noting that for any $x \in \langle H, N \rangle$ there exists $u \in H$ such that $(x, u) \in P_N$, we see that $\langle H, N \rangle / P_N$ is isomorphic with $H/P_{H \cap N}$.

Theorem 6. *If A is an n-unitary subsemigroup of a semigroup S and $M \in U(S)$, then $A \cap M$ is an n-unitary subsemigroup of M and $\langle A, M \rangle / P_A$ is isomorphic with $M/P_{A \cap M}$.*

Proof. By Lemma 1, we may assume that $A \not\subseteq M$. Since A is an n-unitary subsemigroup of S, it follows that $\langle A, M \rangle = AM \in U(S)$. So, for every $x \in A$, there are elements $a \in A$ and $m \in M$ such that $x = am$. Since A is unitary in S, it follows that $m \in A$, that is $A \cap M \neq \phi$. We prove that $A \cap M$ is an n-unitary subsemigroup of M.

Evidently, $A \cap M \in U(M)$. To prove that $A \cap M$ is reflexive in M, let a and b be arbitrary elements in M with $ab \in A \cap M$. Then $ab \in A$ and $ab \in M$, which imply that $ba \in A \cap M$, because A is reflexive in S and M is a subsemigroup of S. So $A \cap M$ is reflexive in M.

Let $V \in U(M)$ arbitrary. Since

$$\begin{aligned} M \cap AV &= M \cap [(A \cap M)V \cup (A - M)V] \\ &= [M \cap (A \cap M)V] \cup [M \cap (A - M)V], \end{aligned}$$

we have

$$M \cap AV = M \cap (A \cap M)V = (A \cap M)V,$$

because

$$M \cap (A - M)V = \phi.$$

Since A is an n-unitary subsemigroup of S and $V \in U(S)$, it follows that $\langle A, V \rangle = AV \in U(S)$. Consequently, $M \cap AV \in U(M)$, that is $(A \cap M)V \in U(M)$. Since

$$(A \cap M)V \subseteq \langle A \cap M, V \rangle$$

and

$$\langle A \cap M, V \rangle \subseteq \langle A, V \rangle \cap \langle M, V \rangle = AV \cap M = (A \cap M)V,$$

we get

$$\langle A \cap M, V \rangle = (A \cap M)V = M \cap AV \in U(M).$$

Thus $A \cap M$ is an n-unitary subsemigroup of M. The isomorphism between $\langle A, M \rangle / P_A$ and $M/P_{A \cap M}$ follows from Lemma 5. Thus the theorem is proved.

References

[1] Clifford, A. H., and G. B. Preston: The algebraic theory of semigroups. Vols. I–II, Amer. Math. Soc., Providence R. I. (1961, 1967).

[2] Nagy, A.: A generalization of the Jordan-Hölder theorem (to appear in Semigroup Forum).