28. An Algebraic Computation of the Alexander Polynomial of a Plane Algebraic Curve

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1. Introduction and the statement of the result. Let C be the algebraic curve in $\boldsymbol{C}^2$ defined by a reduced polynomial f. We denote by $\Omega^{\cdot}_{C^2}(*C)$ the algebra of the rational differential forms on $\boldsymbol{C}^2$ which are holomorphic on the complement $X=\boldsymbol{C}^2-C$. Let $\nabla_\alpha : \Omega^j_{C^2}(*C) \to \Omega^{j+1}_{C^2}(*C)$ be the regular connection in the sense of Deligne ([1]) defined by $\nabla_\alpha \varphi = d\varphi + \alpha d \log f \wedge \varphi$ with a real number α. We denote by $H^j(\Omega^{\cdot}_{C^2}(*C), \nabla_\alpha)$ the j-th cohomology group of the de Rham complex

$$\cdots \longrightarrow \Omega^j_{C^2}(*C) \xrightarrow{\nabla_\alpha} \Omega^{j+1}_{C^2}(*C) \longrightarrow \cdots.$$

In [5] A. Libgober defined the Alexander polynomial of a plane algebraic curve. Let us review the definition.

Definition 1.1. Let $\bar{C}$ be an irreducible algebraic curve in $\boldsymbol{P}^2$. We take a complex line H_∞ such that H_∞ intersects $\bar{C}$ transversally. Let C denote $\bar{C} \cap (\boldsymbol{P}^2 - H_\infty)$ and let X be the complement of C in $\boldsymbol{P}^2 - H_\infty$.

Let $p: X^{ab} \to X$ be an infinite cyclic covering of X. Then the ring of the Laurent polynomials $\boldsymbol{C}[t^{-1}, t] = \Lambda$ operates on $H^1(X^{ab}; \boldsymbol{C})$ by means of the deck transformations. The Λ-module $H^1(X^{ab}; \boldsymbol{C})$ has a presentation of the form

$$\Lambda/_{(f_1(t))} \oplus \cdots \oplus \Lambda/_{(f_k(t))}$$

with some polynomials $f_1(t), \cdots, f_k(t)$. We call the product $\prod_{j=1}^k f_j(t)$ the *Alexander polynomial* of $\bar{C}$ (or C).

Remarks 1.2. i) In the proof of Theorem (1.3), we show that $\dim_C H^1(X^{ab}; \boldsymbol{C})$ is finite.

ii) The Alexander polynomial of the curve C is determined up to unit and does not depend on the choice of a line H_∞.

We have the following

Theorem 1.3. *Let $C \cap \boldsymbol{C}^2$ be an irreducible algebraic curve which intersects transversally with the line at infinity. Let h_α denote $\dim_C H^1(\Omega^{\cdot}_{C^2}(*C, \nabla_\alpha)$ for a real number α. Let $\Delta_C(t)$ be the Alexander polynomial of C. Then we have*

$$\Delta_C(t) = \prod_{0<\alpha<1} (t - \exp 2\pi\sqrt{-1}\,\alpha)^{h_\alpha}.$$

Moreover the numbers α with $h_\alpha \neq 0$ are rational numbers with $n\alpha \in \boldsymbol{Z}$, where we denote by n the degree of our curve C.

2. Proof of the theorem. Let $\bar{C}$ be the algebraic closure of C

in $\boldsymbol{P}^2$ and let X_n be the n-fold cyclic covering of $X=C^2-C$ defined by $X_n=\{(x_1, x_2, x_3)\in \boldsymbol{C}^3 ; x_3^n=f(x_1, x_2), x_3\neq 0\}$. Let $\overline{X}_n$ be the algebraic closure of X_n in $\boldsymbol{P}^3$. Thus, we obtain the branched covering $\pi:\overline{X}_n\to \boldsymbol{P}^2$ branched along $\overline{C}$. Let $\mu: V\to\overline{X}_n$ be a resolution of singularity. We put $D=\mu^{-1}(\pi^{-1}(\overline{C}))$. Let $i:\overline{X}_n-\pi^{-1}(\overline{C})\to V$ be the injection defined by $i=(\mu|_{V-D})^{-1}$. We have the following proposition due to R. Randell.

Proposition 2.1 (Randell [7]). *The injection* $i:\overline{X}_n-\pi^{-1}(\overline{C})\to V$ *induces an isomorphism* $i^*: H^1(V;\boldsymbol{Q})\cong H^1(\overline{X}_n-\pi^{-1}(\overline{C});\boldsymbol{Q})$.

Let γ be a generator of the group of the deck transformations of the cyclic covering $p_n: X_n\to X$. We denote by $H^1(X_n;\boldsymbol{C})_k$ the subspace of $H^1(X_n;\boldsymbol{C})$ defined by

$$H^1(X;\boldsymbol{C})_k=\{\omega\in H^1(X_n;\boldsymbol{C}) ; \gamma^*\omega=\exp 2\pi\sqrt{-1}(k/n)\}.$$

Lemma 2.2. *Let* $j: X_n\hookrightarrow\overline{X}_n-\pi^{-1}(\overline{C})$ *be the inclusion. Then, the induced homomorphism* $j^*: H^1(\overline{X}_n-\pi^{-1}(\overline{C});\boldsymbol{C})$ *is injective and* image $j^*=\bigoplus_{0<k\leq n-1} H^1(X_n;\boldsymbol{C})_k$.

Proof. Let H_∞ be the line at infinity. From the cohomology exact sequence of the pair $(\overline{X}_n-\pi^{-1}(\overline{C}), X_n)$ and the Thom isomorphism $H^k(\overline{X}_n-\pi^{-1}(\overline{C}), X_n)\cong H^{k-2}(\pi^{-1}(H_\infty-\overline{C}))$, we have the injectivity of j^*. We obtain the exact sequence

$$0\longrightarrow H^1(\overline{X}_n-\pi^{-1}(\overline{C}))\xrightarrow{j^*}H^1(X_n)\longrightarrow \underset{\boldsymbol{C}}{\underset{\wr\Vert}{H^2(\overline{X}_n-\pi^{-1}(\overline{C}), X_n)}}\longrightarrow 0.$$

On the other hand $H^1(X_n;\boldsymbol{C})_0=\boldsymbol{C}$, which is represented by $d\log f$. This completes the proof of the second assertion.

Let $\varpi: X^{ab}\to X_n$ be the covering map such that $p_n\circ\varpi=p$.

Lemma 2.3. *The covering map* ϖ *induces an injection*

$$\varpi^*: \bigoplus_{1\leq k\leq n-1} H^1(X_n;\boldsymbol{C})_k\longrightarrow H^1(X^{ab};\boldsymbol{C}).$$

Proof. Let $[\omega]$ be an element of $H^1(X_n;\boldsymbol{C})_k$ such that $\varpi^*\omega=df$ for some function f on X^{ab}. We have $\gamma^*f=\exp 2\pi\sqrt{-1}(k/n)f+c$ for some constant c. If $k\neq 0$, we put $g=f+c(\exp 2\pi\sqrt{-1}(k/n)-1)^{-1}$ which satisfies $dg=\varpi^*\omega$.

We put $\tilde{C}=\pi^{-1}(C)$. Let $\Omega^\cdot(*\tilde{C})$ be the algebra of rational 1-forms on the surface $x_3^n=f(x_1, x_2)$ which have poles at most along $\tilde{C}$. Let $\Omega^\cdot(*\tilde{C})[k/n]$ be the vector space of the rational forms $\varphi\in\Omega^\cdot(*\tilde{C})$ such that $\gamma^*\varphi=\exp 2\pi\sqrt{-1}(k/n)\varphi$. By means of the comparison theorem of Grothendieck-Deligne [2], we have the following isomorphism

$$H^j(\Omega^\cdot(*\tilde{C})[k/n]\cong H^j(X_n;\boldsymbol{C})_k. \tag{2.4}$$

On the other hand, we have the following commutative diagram.

$$\begin{array}{ccc} \Omega^j(*\tilde{C})[\alpha] & \xrightarrow{d} & \Omega^{j+1}(*\tilde{C})[\alpha] \\ \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\varphi} \\ \Omega^j_{C^2}(*C) & \xrightarrow{\nabla_\alpha} & \Omega^{j+1}_{C^2}(*C) \end{array} \tag{2.5}$$

where the homomorphism φ is defined by $\varphi(\omega)=f^{\alpha}\omega$ for $\omega\in\Omega^{j}_{C^2}(*C)$. This homomorphism induces the following isomorphism

$$H^{j}(\Omega^{\cdot}_{C^2}(*C),\nabla_{\alpha})\cong H^{j}(X_n;C)_{\alpha}.$$

Combining (2.1)–(2.5), we get the following commutative diagram.

$$(2.6)\qquad \begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ & & H^1(\overline{X}_n-\pi^{-1}(\overline{C});C) & \xleftarrow[i^*]{\cong} & H^1(V;C) \\ & & \downarrow j^* & & \\ 0\longrightarrow H^1(X_n;C)^{\pi_1} & \longrightarrow & H^1(X_n;C) & \xrightarrow[\varpi^*]{} & H^1(X^{ab};C) \\ \uparrow \wr\wr & & \uparrow \wr\wr & & \\ 0\longrightarrow H^1(X;C) & \longrightarrow & \bigoplus_{0\le k\le n-1} H^1(\Omega^{\cdot}_{C^2}(*C),\nabla_{k/n}) & & \end{array}$$

Proposition 2.7. *The homomorphism*

$$\varpi^*\circ j^*\circ i^*: H^1(V;C)\longrightarrow H^1(X^{ab};C)$$

is an isomorphism.

Proof. From (2.1)–(2.3), the homomorphism $\varpi^*\circ j^*\circ i^*$ is injective. Let q be the irregularity of V. It suffices to prove that $\dim H^1(X^{ab};C)=2q$.

Since $\overline{C}$ intersects H_{∞} transversally, we have the following exact sequence of the central extension ([6], Lemma 1).

$$0\longrightarrow Z\longrightarrow\pi_1(C^2-C,*)\longrightarrow\pi_1(P^2-\overline{C},*)\longrightarrow 1.$$

Hence, the following isomorphism follows

$$[\pi_1(C^2-C,*),\pi_1(C^2-C,*)]\cong[\pi_1(P^2-\overline{C},*),\pi_1(P^2-\overline{C},*)].$$

By using Proposition 2.1, we have $\dim H^1(X^{ab};C)=2q$.

Thus, we have the following isomorphisms.

$$H^1(X^{ab};C)\cong\bigoplus_{0<k\le n-1}H^1(X_n;C)_k\cong\bigoplus_{0<k\le n-1}H^1(\Omega^{\cdot}_{C^2}(*C),\nabla_{k/n}).$$

This completes the proof of Theorem 1.3.

3. Discussion and examples. Let ρ_{α} be the representation of $\pi_1(C^2-C,*)$ defined by

$$\begin{array}{l} \rho_{\alpha}:\pi_1(C^2-C,*)\longrightarrow H_1(C^2-C;Z)\longrightarrow C^* \\ \qquad\qquad\qquad\qquad\qquad\qquad \wr\wr \qquad\qquad\quad \cup \\ \qquad\qquad\qquad\qquad\qquad\qquad Z\ni 1\longmapsto\exp 2\pi\sqrt{-1}\,\alpha. \end{array}$$

We denote by $\mathscr{V}(\alpha)$ the flat vector bundle associated with the representation ρ_{α}. We have an isomorphism

$$H^{j}(\Omega^{\cdot}_{C^2}(*C),\nabla_{\alpha})\cong H^{j}(X;\mathscr{V}(\alpha)).$$

Hence, our theorom can also be formulated by using $\mathscr{V}(\alpha)$.

In [4] we prove that $h_{\alpha}=0$ if $\exp 2\pi\sqrt{-1}\,\alpha$ is not one of the eigenvalues of the Milnor monodromies at the singular points of C. In particular, if we assume that C posseses only cusps and nodes as singularities, $\bigoplus_{0<\alpha<1}H^1(\Omega^{\cdot}_{C^2}(*C),\nabla_{\alpha})$ is ismorphic to

$$H^1(\Omega^{\cdot}_{C^2}(*C),\nabla_{1/6})\oplus H^1(\Omega^{\cdot}_{C^2}(*C),\nabla_{-1/6}).$$

The Alexander polynomial of C is $(t^2-t+1)^q$, where q is the irregularity in the sense of the previous section. We have

$$q \leqslant \text{the number of cusps.}$$

For the proof of these statements see [4].

In [10], Zariski studies the irregularity by means of linear systems. In our point of view $q=\dim H^1(\Omega^{\cdot}_{C^2}(*C), \nabla_{1/6})$.

Let $\chi(X)$ be the Euler characteristic of X. We have

$$\dim H^1(\Omega^{\cdot}_{C^2}(*C), \nabla_\alpha) - \dim H^2(\Omega^{\cdot}_{C^2}(*C), \nabla_\alpha) = \chi(X) \qquad \text{for any } \alpha.$$

In particular, if we assume that C has only cusps and nodes as singularities, we have

$$\dim H^2(\Omega^{\cdot}_{C^2}(*C), \nabla_\alpha) = \begin{cases} \chi(X) & \text{if } \alpha \equiv \pm 1/6 \bmod . \, Z \\ \chi(X)+q & \text{otherwise.} \end{cases}$$

Example 3.1. Let X be the complement of the curve defined by $x^2-y^3=0$ in C^2. We have $\dim H^1(X^{ab}; C)=2$ and $H^1(X^{ab}; C)$ is isomorphic to

$$H^1(\Omega^{\cdot}_{C^2}(*C), \nabla_{1/6}) \oplus H^1(\Omega^{\cdot}_{C^2}(*C), \nabla_{-1/6})$$

and is represented by the differential forms $\omega_1=(x^2-y^3)^{-1}(-(y/3)dx+(x/2)dy)$, $\omega_2=y\omega_1$.

Example 3.2. Let C be the curve defined by

$$f=(x^2+y^2)^3+(y^3+1)^2=0 \qquad \text{(see [9])}.$$

Then, $H^1(\Omega^{\cdot}_{C^2}(*C), \nabla_{1/6})$ and $H^1(\Omega^{\cdot}_{C^2}(*C), \nabla_{-1/6})$ have dimension 1 and are generated respectively by $\omega_1=f^{-1}(x(y^3+1)dx+(y(y^3+1)-y^2(x^2+y^2))dy)$ and $\omega_2=(y^3+1)\omega_1$. The Alexander polynomial is t^2-t+1.

References

[1] Deligne, P.: Equations différentielles à points singuliers réguliers III. Lect. Note in Math., vol. 163, Springer (1970).

[2] Grothendieck, A.: On the de Rham cohomology of algebraic varieties. Publ. Math. I.H.E.S., **29**, 95–103 (1966).

[3] Kohno, T.: Differential forms and the fundamental group of the complement of hypersurfaces (to appear in Proc. of pure Math., A.M.S.).

[4] ——: Etude algébrique du groupe fondamental du complément des hypersurfaces et problèmes de $K(\pi, 1)$. Thèse de 3ème Cycle, Université Paris VII (1982).

[5] Libgober, A.: Alexander polynomials of plane algebraic curves (1981) (preprint).

[6] Oka, M.: On the fundamental group of the complement of a reducible curve in P^2. J. London Math. Soc., **12**, 239–252 (1976).

[7] Randell, R.: Some topology of Zariski surfaces. Lect. Note in Math., vol. 788, Springer, pp. 145–165 (1979).

[8] Sullivan, D.: Infinitesimal computations in topology. Publ. Math. I.H.E.S., **47**, 269–331 (1977).

[9] Zariski, O.: On the existence of algebraic functions of two variables possessing a given branched curve. Amer. J. Math., **51**, 305–328 (1929).

[10] ——: On the irregularity of cyclic multiple planes. Ann. of Math., **32**, 131–141 (1931).