132. *On Poles of the Rational Solution of the Toda Equation of Painlevé-IV Type*

By Yoshinori KAMETAKA
Department of Applied Mathematics, Faculty of Engineering, Ehime University

(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 12, 1983)

§1. **Okamoto's polynomials.** K. Okamoto [1] found an interesting new rational solution of the Toda equation which produces a sequence of infinitely many rational solutions of Painlevé-IV equation. According to him the recurrence relations

(1.1) $$P_0=1, \qquad P_1=t,$$

(1.2) $$P_{n-1}P_{n+1}=P_nP_n''-P_n'^2+(t^2-2n)P_n^2,$$

(1.3) $$P_{-n}(t)=i^{n(n+2)}P_n(it),$$

(1.4) $$Q_0=1, \qquad Q_1=t^2-1,$$

(1.5) $$Q_{n-1}Q_{n+1}=Q_nQ_n''-Q_n'^2+(t^2-2n-1)Q_n^2,$$

(1.6) $$Q_{-n}(t)=i^{n(n-1)}Q_{n-1}(it), \qquad n=1,2,3,\cdots$$

determine two series of polynomials

(1.7) $$P_n=\sum_{j=0}^{[n^2/2]} P_{n,j}t^{n^2-2j},$$

(1.8) $$Q_n=\sum_{j=0}^{n(n+1)/2} Q_{n,j}t^{n(n+1)-2j}$$

with integral coefficients ($P_{n,0}=Q_{n,0}=1$).

(1.9) $$r_n=P_{n-1}P_{n+1}/P_n^2=(\log P_n)''+t^2-2n,$$

(1.10) $$s_n=(\log P_{n-1}/P_n)'+2t$$

satisfies the Toda equation

(1.11) $$s_n'=r_{n-1}-r_n, \qquad r_n'=r_n(s_n-s_{n+1}).$$

(1.12) $$\tilde{r}_n=Q_{n-1}Q_{n+1}/Q_n^2=(\log Q_n)''+t^2-2n-1,$$

(1.13) $$\tilde{s}_n=(\log Q_{n-1}/Q_n)'+2t$$

also satisfies the Toda equation

(1.14) $$\tilde{s}_n'=\tilde{r}_{n-1}-\tilde{r}_n, \qquad \tilde{r}_n'=\tilde{r}_n(\tilde{s}_n-\tilde{s}_{n+1}).$$

If we define q_n and p_n by

(1.15) $$q_n=-P_{n+1}Q_{n-1}/P_nQ_n,$$

(1.16) $$p_n=-P_{n-1}Q_n/P_nQ_{n-1}=-iq_{-n}(it)$$

then

(1.17) $$y_n(x)=\sqrt{2/3}\,q_n(\sqrt{2/3}\,x)$$

satisfies P-IV $(n,\ -2(n+1/3)^2)$ where we mean by P-IV (α,β) the Painlevé-IV equation

(1.18) $$y''=y'^2/2y+(3/2)y^3+4xy^2+2(x^2-\alpha)y+\beta/y.$$

Notice that

(1.19) $$z_n(x)=\sqrt{2/3}\,p_n(\sqrt{2/3}\,x)=-iy_{-n}(ix)$$
also satisfies Painlevé-IV equation P-IV $(n, -2(n-1/3)^2)$.

§ 2. **Main results.** We proved that

Theorem 2.1. (1) *P_n and Q_n are really polynomials of degree n^2 and $n(n+1)$ with integral coefficients. Q_n are even functions, that is, are polynomials of t^2 of degree $n(n+1)/2$.*

(2) *All zeros of P_n and Q_n are simple.*

(3) *Each pair of polynomials $\{P_n, P_{n+1}\}$, $\{Q_n, Q_{n+1}\}$, $\{P_n, Q_n\}$, $\{P_{n+1}, Q_n\}$ and $\{P_{n+2}, Q_n\}$ has no common zero.*

(4) *r_n ($\tilde{r}_n$) has n^2 ($n(n+1)$) double poles.*

(5) *s_n ($\tilde{s}_n$) has $2n(n-1)+1$ ($2n^2$) simple poles.*

(6) *q_n (p_n) has $n(2n+1)$ ($n(2n-1)$) simple poles.*

As a consequence of the above theorem P_n and Q_n can be expressed as

(2.1) $$P_n=\prod_{k=1}^{n^2}(t-a_{n,k}), \qquad Q_n=\prod_{k=1}^{n(n+1)}(t-b_{n,k}).$$

Sharp estimates for the maximal moduli

(2.2) $$A_n=\max\{|a_{n,k}|;\ 1\le k\le n^2\},$$

(2.3) $$B_n=\max\{|b_{n,k}|;\ 1\le k\le n(n+1)\}$$

for zeros of these polynomials were obtained.

Theorem 2.2 (Main theorem).

(2.4) $$\{2n(n+2)/3(n+1)\}^{1/2}\le A_{n+1}\le 3n^{1/2},$$

(2.5) $$\{(2n+3)/3\}^{1/2}\le B_{n+1}\le 3\{(2n+1)/2\}^{1/2}, \qquad n=0, 1, 2, \cdots.$$

Moreover we can show the inequality

(2.6) $$B_{n+1}>A_{n+1}>B_n>A_n>B_1=1>A_1=0, \qquad n=2, 3, 4, \cdots.$$

The proof of our main theorem is almost the same as that for our previous result [2]. We showed an analogous sharp estimate for the maximal modulus of poles of the rational solution of the Toda equation of Painlevé-II type. Detailed proof will be published elsewhere. Here we only list up the fundamental recurrence relations which are satisfied by rational functions q_n, p_n, r_n, s_n, $\tilde{r}_n$ and $\tilde{s}_n$.

§ 3. **Recurrence relations.** The rational functions q_n and p_n are uniquely determined by the recurrence relation

(3.1) $$p_0=q_0=-t,$$

(3.2) $$p_n=-p_{n-1}-q_{n-1}-3t-(3n-2)/q_{n-1},$$

(3.3) $$q_n=-p_n-q_{n-1}-3t-(3n-1)/p_n,$$

(3.4) $$p_{-n}(t)=-iq_n(it), \quad q_{-n}(t)=-ip_n(it), \quad n=1, 2, 3, \cdots.$$

We can derive the following relations

(3.5) $$p'_n=p_n(p_n+2q_n+3t)+3n-1,$$

(3.6) $$q'_n=-q_n(2p_n+q_n+3t)-3n-1.$$

Eliminating p_n from (3.5) and (3.6) we can show that $y_n(x)$ defined by (1.17) satisfies Painlevé-IV equation. Above relations (3.5) and (3.6) can also be expressed as

(3.7) $$p_n'=p_n(q_n-q_{n-1}), \qquad q_n'=q_n(p_{n+1}-p_n).$$

So if we introduce

(3.8) $$r_n=p_nq_n, \qquad s_n=-p_n-q_{n-1}$$

and

(3.9) $$\tilde{r}_n=p_{n+1}q_n, \qquad \tilde{s}_n=-p_n-q_n$$

then $\{r_n, s_n\}$ and $\{\tilde{r}_n, \tilde{s}_n\}$ are both solutions of the Toda equation. Values of these rational solutions can be calculated through the following recurrence relations.

(3.10) $$s_0=2t+t^{-1}, \qquad r_0=t^2,$$

(3.11) $$s_n=\{(r_{n-1}+3n-4)(r_{n-1}+3n-2)\}/\{r_{n-1}(s_{n-1}-3t)\}+3t,$$

(3.12) $$r_n=-r_{n-1}-6n+3-s_n(s_n-3t),$$

(3.13) $$r_{-n}(t)=-r_n(it), \quad s_{-n}(t)=-is_{n+1}(it), \quad n=1,2,3,\cdots,$$

(3.14) $$\tilde{s}_0=2t, \qquad \tilde{r}_0=t^2-1,$$

(3.15) $$\tilde{s}_n=\{(\tilde{r}_{n-1}+3n-1)(\tilde{r}_{n-1}+3n-2)\}/\{\tilde{r}_{n-1}(\tilde{s}_{n-1}-3t)\}+3t,$$

(3.16) $$\tilde{r}_n=-\tilde{r}_{n-1}-6n-\tilde{s}_n(\tilde{s}_n-3t),$$

(3.17) $$\tilde{r}_{-n}(t)=-\tilde{r}_{n-1}(it), \quad \tilde{s}_{-n}(t)=-i\tilde{s}_n(it), \quad n=1,2,3,\cdots.$$

References

[1] K. Okamoto: private communication.
[2] Y. Kametaka: On poles of the rational solution of the Toda equation of Painlevé-II type. Proc. Japan Acad., **59**A, 358–360 (1983).