107. A Truncated Cube Functional Equation

By Shigeru HARUKI
Okayama University of Science

(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1983)

§1. **Introduction.** The purpose of this note is to announce some equivalence relations among certain particular polyhedral mean value type functional equations without any regularity assumptions.

Let $(G, +)$ be an Abelian group in which it is possible to divide by 2, and let F be a field of characteristic zero. For a function $f: G\times G\times G\to F$ we define the shift operators X_1^t, X_2^t, and X_3^t by $(X_1^t f)(x, y, z)=f(x+t, y, z)$, $(X_2^t f)(x, y, z)=f(x, y+t, z)$, and $(X_3^t f)(x, y, z)=f(x, y, z+t)$ for all $x, y, z, t\in G$. In particular $1=X_1^0=X_2^0=X_3^0$ denotes the identity operator. We note that the ring of linear transformation generated by this family of transformations is commutative and distributive.

L. Etigson [2] and L. Sweet [5] considered the equivalence of the following cube and octahedron mean value functional equations, which are the most fundamental particular polyhedral mean value type functional equations, under the assumption $f: G\times G\times G\to F$:

(1.1) $$(C(t)f)(x, y, z)=8f(x, y, z),$$

(1.2) $$(O(t)f)(x, y, z)=6f(x, y, z)$$

where the operators $C(t)$ and $O(t)$ are defined by

$$C(t)=\prod_{i=1}^{3}(X_i^t+X_i^{-t}) \quad \text{and} \quad O(t)=\sum_{i=1}^{3}(X_i^t+X_i^{-t}).$$

In this note we will consider the equivalence of (1.1) and the polyhedral mean value functional equation

(1.3) $$(T(t)f)(x, y, z)=12f(x, y, z)$$

where the operator $T(t)$ is defined by

$$T(t)=(X_1^t+X_1^{-t})(X_2^t+X_2^{-t})+(X_2^t+X_2^{-t})(X_3^t+X_3^{-t})+(X_3^t+X_3^{-t})(X_1^t+X_1^{-t}).$$

By a geometric interpretation we call equation (1.3) a *truncated cube mean value functional equation.*

§2. **Equivalence of (1.1) and (1.3).** **Theorem 1.** *If a function $f: G\times G\times G\to F$ satisfies equation* (1.1) *for all $x, y, z, t\in G$, then also* (1.3) *for all $x, y, z, t\in G$ and conversely so that* (1.1) *and* (1.3) *are equivalent.*

By using the operator notations in §1 we have $C(2t)=\prod(X_i^{2t}+X_i^{-2t})$ and readily obtain

(i) $C(t)^2=(C(t))(C(t))=C(2t)+2T(2t)+4O(2t)+8,$

(ii) $O(t)^2=(O(t))(O(t))=O(2t)+2T(t)+6,$

(iii) $T(t)^2=(T(t))(T(t))=T(2t)+4O(2t)+2O(t)C(t)+12.$

Proof of Theorem 1. We briefly write (1.3) as

$$T(t)=12. \tag{2.4}$$

Squaring the operators on both sides of (2.4) yields $T(2t)+4O(2t)+2O(t)C(t)+12=144$ which, with (2.4), implies

$$O(t)C(t)=60-2O(2t). \tag{2.5}$$

It follows from (i), (ii), and (2.4) that

$$C(t)^2=C(2t)+4O(2t)+32, \qquad O(t)^2=O(2t)+30. \tag{2.6}$$

Now, square both sides of (2.5) and then use (2.6) to obtain $(O(2t)+30)(C(2t)+4O(2t)+32)=3600-240O(2t)+4O(2t)^2$ or, in expanded form, $O(2t)C(2t)+4O(2t)^2+32O(2t)+30C(2t)+120O(2t)+960=3600-240O(2t)+4O(2t)^2$, which, with (2.5) implies $60-2O(4t)+392O(2t)+30C(2t)=2640$, $30C(2t)+392O(2t)=2O(4t)+2580$, and $15C(2t)+196O(2t)=O(4t)+1290$. By replacing $2t$ by t we have

$$15C(t)+196O(t)=O(2t)+1290. \tag{2.7}$$

Squaring both sides of (2.7) yields $225C(2t)+900O(2t)+7200+5880O(t)C(t)+38416O(2t)+1152480=O(4t)+30+2580O(2t)+1664100$ which, with (2.5), implies $225C(2t)+900O(2t)+7200+352800-11760O(2t)+38416O(2t)+1152480=O(4t)+30+2580O(2t)+1664100$ and $-O(4t)+225C(2t)+24976O(2t)=151650$. By replacing $2t$ by t this equation becomes

$$225C(t)+24976O(t)=O(2t)+151650. \tag{2.8}$$

Next, multiply both sides of (2.7) by 15 to obtain

$$225C(t)+2940O(t)=15O(2t)+19350. \tag{2.9}$$

Subtract (2.9) from (2.8) to obtain $14O(2t)+22036O(t)=132300$ and

$$O(2t)+1574O(t)=9450. \tag{2.10}$$

Thus (2.4) implies (2.10). Write (2.10) as

$$1574O(t)=9450-O(2t) \tag{2.11}$$

and then square both sides of (2.11) to obtain $2477476O(2t)+74324280=89302500-18900O(2t)+O(4t)+30$ and $O(4t)-2496376O(2t)=-14978250$ which, with $2t$ replaced by t, implies

$$O(2t)-2496376O(t)=-14978250. \tag{2.12}$$

Subtract (2.10) from (2.12) to obtain $-2497950O(t)=-14987700$ and $O(t)=6$. Now it follows from (2.5) and $O(2t)=6$ that $6C(t)=60-12$ and $C(t)=8$, that is, (1.1). Thus (1.3) implies (1.1).

Conversely, squaring both sides of $C(t)=8$, we have $C(2t)+2T(2t)+4O(2t)+8=64$, or, with $2t$ replaced by t,

$$C(t)+2T(t)+4O(t)+8=64. \tag{2.13}$$

On the other hand, by a result of [1] or [2], $C(t)=8$ implies $O(t)=6$. Hence, it follows from (2.13), $C(t)=8$, and $O(t)=6$ that $8+2T(t)+24+8=64$ and $T(t)=12$, that is, (2.4). Thus (1.1) and (1.3) are equivalent.

§ 3. Consequences of Theorem 1. Let R be the set of all real numbers. Then by combining results of H. Haruki [3] and M. A. McKiernan [6] (see also [4]) with $G=F=R$ we obtain the following two corollaries.

Corollary 1. *If $f: R\times R\times R\to R$ is bounded on a set of positive Lebesgue measure and is a solution of* (1.3), *then $f\in C^\infty$.*

Corollary 2. *The only solution $f: R\times R\times R\to R$ of* (1.3) *which is bounded on a set of positive Lebesgue measure is given by*

$$f(x,y,z)=\sum_{i,j,k=0}^{5}\alpha_{ijk}(\partial^{i+j+k}P(x,y,z))/(\partial x^i\partial y^j\partial z^k) \tag{3.14}$$

where $\{\alpha\}_{ijk}$ are real constants and

$$P(x,y,z)=xyz(x^2-y^2)(y^2-z^2)(z^2-x^2).$$

(3.14) *is also the only continuous solution.*

§ 4. A related equation. Theorem 2. *If a function $f: R\times R\times R\to R$ satisfies equation* (1.1) *for all $x, y, z, t\in G$, then also*

$$((C(t)+O(t)+T(t))f)(x,y,z)=26f(x,y,z) \tag{4.15}$$

for all $x, y, z, t\in G$ and conversely so that (1.1) *and* (4.15) *are equivalent.*

A proof of Theorem 2 is similar to that of Theorem 1. We omit it.

§ 5. Conclusion. Theorem 3. *If $f: R\times R\times R\to R$ satisfies the cube mean value functional equation* (1.1) *for all $x, y, z, t\in G$, then also each one of* (1.2), (1.3), *and* (4.15) *for all $x, y, z, t\in G$ and conversely so that they are equivalent to each other.*

References

[1] J. Aczél, H. Haruki, M. A. McKiernan, and G. N. Sakovič: Aequationes Math., **1**, 37–53 (1968).

[2] L. Etigson: ibid., **10**, 50–56 (1974).

[3] H. Haruki: ibid., **3**, 156–159 (1969).

[4] S. Haruki: Utilitas Math., **4**, 3–7 (1973).

[5] L. Sweet: Aequationes Math., **22**, 29–38 (1981).

[6] M. A. McKiernan: ibid., **4**, 31–36 (1968).