102. On Pluri-Genera and Mixed Hodge Structures of Isolated Singularities

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In this article, we show that an isolated hypersurface singularity with the pluri-genera $\delta_m \leq 1$ for any m has good properties in the aspect of mixed Hodge structures. To be precise, it is shown that, for an isolated hypersurface singularity, the three conditions "cohomologically insignificant" "Du Bois" and "$\delta_m \leq 1$ for any m" are equivalent.

First, we introduce these three concepts.

Definition 1. Let $D=\{z \in C \mid |z|<1\}$ be the unit disk, $f: X \to D$ a proper surjective holomorphic map of a connected complex manifold X. We say that $f: X \to D$ is a projective smoothing of X_0, if all fibers $X_t=f^{-1}(t)$ are connected projective algebraic varieties, non-singular for $t \neq 0$.

Definition 2. Let $f: X \to D$ be a projective smoothing. We say that f is a cohomologically insignificant smoothing if the specialization map $sp_i: H^i(X_0) \to H^i(X_\infty)$ induces the isomorphisms of $(p, 0)$-components $H_i^{p,0}(X_0) \simeq H_i^{p,0}(X_\infty)$ for any $p \geq 0$ and $i \geq 0$ (cf. [1]).

Definition 3. Let $x \in Y$ be an isolated hypersurface singularity. We say that $x \in Y$ is a cohomologically insignificant singularity if there are a projective compactification X_0 of Y with only one singular point x and a projective smoothing of X_0 which is cohomologically insignificant.

Proposition 1. (1) *An isolated hypersurface singularity $x \in Y$ is cohomologically insignificant if and only if, for any projective compactification X_0 of Y with x as only one singular point, any projective smoothing of X_0 is cohomologically insignificant.*

(2) *Let X_0 be a projective variety which is non-singular except for isolated hypersurface singularities $x_1, \cdots, x_r$.*

Then any smoothing of X_0 is cohomologically insignificant if and only if $x_1, \cdots, x_r$ are all cohomologically insignificant.

Proof. Proposition 1 follows from the following proposition which asserts that cohomologically insignificance is a local property of the singularity and does not depend on the choice of smoothings.

Proposition 2. *Let $x \in X$ be an isolated hypersurface singularity of dimension $n \geq 2$. Let $\pi: \tilde{X} \to X$ be a good resolution*; i.e. *a resolu-*

tion of the singularity such that $E=\pi^{-1}(x)_{red}$ *is a divisor with simply normal crossings on* $\tilde{X}$. *Denote the dimension of* $(p,0)$*-component of* $Gr_p^W H^i(E)$ *by* $h_i^{p,0}(E)$.

Then,

$$p_g(x)-(-1)^{n-1}\sum_{i=1}^{n-1}(-1)^i\sum_{p=0}^{i}h_i^{p,0}(E)\geq 0,$$

where the equality holds if and only if $x\in X$ *is cohomologically insignificant.*

Definition 4. Let $(\underline{\Omega}_X^{\cdot},F)$ be the Du Bois' filtered complex of a variety X ([2]). We define $x\in X$ to be a Du Bois singularity if the natural map from $\mathcal{O}_{X,x}$ to $Gr_F^0\underline{\Omega}_{X,x}^{\cdot}$ is a quasi-isomorphism.

Proposition 3 ([3]). *Let* $x\in X$ *be a normal isolated singular point of an* n*-dimensional Stein space* X *with* $n\geq 2$.

Then $x\in X$ *is Du Bois if and only if the canonical map* $H^i(\tilde{X},\mathcal{O}_X)\to H^i(E,\mathcal{O}_E)$ *are isomorphisms for any* $i>0$, *where* $\pi:\tilde{X}\to X$ *is a good resolution.*

Definition 5. Let $x\in X$ be a normal isolated singular point of an n-dimensional Stein space X with $n\geq 2$. We define the pluri-genera $\delta_m(m\in Z, m\geq 1)$;

$$\delta_m(X,x)=\dim\frac{\Gamma(X-\{x\},\mathcal{O}(K))}{L^{2/m}(X-\{x\})}.$$

Proposition 4 ([4]). *Let* $\pi:\tilde{X}\to X$ *be a good resolution of a singularity* $x\in X$ *as in Definition* 5. *Denote* $\pi^{-1}(x)_{red}$ *by* E. *Then* $\delta_m(X,x)$ *is represented as follows;*

$$\delta_m(X,x)=\dim\frac{\Gamma(\tilde{X}-E,\mathcal{O}(mK))}{\Gamma(\tilde{X},\mathcal{O}(mK+(m-1)E))}.$$

Now we will mention the main results.

Theorem I. *Let* $x\in X$ *be an isolated hypersurface singularity of dimension* $n\geq 2$. *Then,* $x\in X$ *is cohomologically insignificant if and only if it is Du Bois.*

Proof. The inequality of Proposition 2 can be replaced by $p_g(x)-h^{n-1}(E,\mathcal{O}_E)\geq 0$. Here, the equality means that $x\in X$ is Du Bois by Proposition 3.

Theorem II. *Let* $x\in X$ *be a normal isolated Gorenstein singularity of dimension* $n\geq 2$. *Then,* $x\in X$ *is Du Bois if and only if* $\delta_m(X,x)\leq 1$ *for all positive* $m\in Z$.

Proof. Note that $\delta_m\leq 1$ means that $\delta_m=0$ for all m or $\delta_m=1$ for all m. At first, any rational singularity (i.e. $\delta_m=0$) is Du Bois by Proposition 3. Next we show that if $x\in X$ is a non-rational Du Bois singularity, then $p_g(x)$ must be 1. Finally, in the case $p_g(x)=1$, the equivalence of Du Bois and $\delta_m(X,x)=1$ for any m is shown. This completes the proof.

References

[1] Dolgachev, I.: Cohomologically insignificant degenerations of algebraic varieties. Compositio Math., **42**, Fasc. 3, 279–313 (1981).
[2] Du Bois, Ph.: Complexe de De Rham filtré d'une variété singulière. Bull. Soc. Math. France, **109**, 41–81 (1981).
[3] Steenbrink, J. H. M.: Smoothings of isolated singularities and mixed Hodge structures (preprint).
[4] Higuchi, T., Yoshinaga, E., and Watanabe, K.: Introduction to Complex Analysis of Several Variables. Morikita Shyuppan (1980) (in Japanese).