101. *Microlocal Study of Sheaves. II*

Constructible Sheaves

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(Communicated by Heisuke HIRONAKA, M. J. A., Oct. 12, 1983)

Introduction. On a real (resp. complex) analytic manifold X, we prove that a complex of sheaves is constructible if and only if it satisfies some finiteness property and if its micro-support [5] is a subanalytic (resp. complex analytic) Lagrangian set. Thus we may study the functorial properties, including contact transformations [6], with our previous results on the micro-support of sheaves. As an application we give a direct image theorem for regular holonomic modules in the non proper case.

1. Let X be a real analytic manifold. We use the same notations as in [6]. In particular $SS(F)$ is the micro-support in T^*X of a complex of sheaves on X. In this note we shall only consider sheaves of vector spaces, in order to simplify the discussion.

Let F be a complex of sheaves on X. We shall say that F is weakly $\boldsymbol{R}$-constructible if there exists a subanalytic stratification such that the restriction of the cohomology groups of F to each stratum is locally constant. We denote by $D^+(X)$ the derived category of complexes of sheaves bounded from below and by $D^+_{w\boldsymbol{R}c}(X)$ the full subcategory consisting of weakly $\boldsymbol{R}$-constructible complexes.

Recall (cf. [2]) that a complex $F \in Ob(D^b(X))$ is said to be $\boldsymbol{R}$-constructible if $F \in Ob(D^+_{w\boldsymbol{R}c}(X))$ and moreover for all $x \in X$, the space $H^j(F)_x$ is finite-dimensional. We denote by $D^b_{\boldsymbol{R}-c}(X)$ the full subcategory of $D^+_{w\boldsymbol{R}c}(X)$ of $\boldsymbol{R}$-constructible complexes.

Theorem 1.1. *Let $F \in Ob(D^+(X))$. The following conditions are equivalent.*

i) $F \in Ob(D^+_{w\boldsymbol{R}c}(X))$.

ii) *$SS(F)$ is contained in a subanalytic and isotropic set of T^*X* (isotropic: There exists a dense open smooth manifold in $SS(F)$ on which the fundamental 1-form vanishes).

iii) *$SS(F)$ is a closed conic Lagrangian subanalytic set of T^*X.*

For the proof we use the technics of [1] and [5], [6].

As a corollary of Theorem 1.1 we prove that if Y is a submanifold of X and $F \in Ob(D^b_{\boldsymbol{R}c}(X))$ then $\nu_Y(F)$ the specialization of F along Y

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belongs to $Ob(D^b_{Rc}(T_Y X))$ and $\mu_Y(F)$ the microlocalization of F along Y (cf. [7]) belongs to $Ob(D^b_{Rc}(T^*_Y X))$. Moreover if φ is an analytic contact transformation from $V \subset T^*Y$ to $U \subset T^*X$ and φ_K a quantized contact transformation over φ (cf [6]), then for $G \in Ob(D^b_{Rc}(Y))$ and $p \in U$, there exists $F \in Ob(D^b_{Rc}(X))$ such that $\varphi_K(G) \simeq F$ in $D^+(X, p)$.

2. Now we assume that X is a complex analytic manifold. We denote by X^R the underlying real manifold, but we often confuse X and X^R. We define the category $D^+_{wCc}(X)$ and $D^b_{Cc}(X)$ as in the real case, but by considering now stratifications by complex manifolds.

Theorem 2.1. *Let $F \in Ob(D^+(X))$. The following conditions are equivalent.*

i) $F \in Ob(D^+_{wCc}(X))$,

ii) $F \in Ob(D^+_{wRc}(X^R))$ *and $SS(F)$ is stable by the action of $C^\times$.*

iii) *$SS(F)$ is contained in a closed conic isotropic subanalytic set of T^*X stable by the action of $C^\times$.*

iv) *$SS(F)$ is a closed conic complex analytic Lagrangian subset of T^*X.*

For a subset $A \subset X$, the conormal cone $N^*(A) \subset T^*X$ is defined in [9]. Let Y be another complex analytic manifold, f a holomorphic map from Y to X. We denote by ρ and $\tilde{\omega}$ the natural associated maps from $Y \underset{X}{\times} T^*X$ to T^*Y and T^*X, respectively.

Theorem 2.3. *Let $(Y_s)_{s>0}$ be a family of open sets in Y^R, let $G \in Ob(D^b_{Cc}(Y))$ and assume:*

i) $Y = \bigcup_s Y_s$, $\bigcup_{s'>s} Y_{s'} = Y_s$ *and* $\bigcap_{s'>s} Y_{s'} \subset \overline{Y}_s$.

ii) *f is proper over* Supp $(G) \cap \overline{Y}_s$ *for all p.*

iii) $N^*Y_s \cap \overline{SS(G) + \rho(Y \underset{X}{\times} T^*X)} \subset T^*_Y Y$.

Then $\boldsymbol{R}f_*(G)$ and $\boldsymbol{R}f_!(G)$ belong to $Ob(D^b_{Cc}(X))$. Moreover $SS(\boldsymbol{R}f_*(G))$ and $SS(\boldsymbol{R}f_!(G))$ are contained in $\tilde{\omega}\rho^{-1}(SS(G))$.

This theorem is easily deduced from Theorem 2.2 and our results in [5].

3. We still consider complex manifolds. Let $\mathcal{D}_X$ denote the sheaf of finite-order holomorphic differential operators on X. For a map $f: Y \to X$ we also consider the $(\mathcal{D}_Y, f^{-1}\mathcal{D}_X)$-bimodule $\mathcal{D}_{Y \to X} = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X$ (cf. [7]).

Let $D(\mathcal{D}_X)$ denote the derived category of the category of complexes of right $\mathcal{D}_X$-modules. We denote by $D^b_{coh}(\mathcal{D}_X)$ (resp. $D^b_{rh}(\mathcal{D}_X)$) the full subcategory of $D(\mathcal{D}_X)$ consisting of complexes whose cohomology is bounded and coherent (resp. bounded and regular holonomic, [3]).

We define two functors from $D^b_{coh}(\mathcal{D}_Y)$ to $D(\mathcal{D}_X)$ by setting for $\mathcal{M} \in Ob(D^b_{coh}(\mathcal{D}_Y))$

$$\int_f \mathcal{N} = \boldsymbol{R}f_*(\mathcal{N} \otimes^L_{\mathcal{D}_Y} \mathcal{D}_{Y \to X})$$
$$\int_f^{pr} \mathcal{N} = \boldsymbol{R}f_!(\mathcal{N} \otimes^L_{\mathcal{D}_Y} \mathcal{D}_{Y \to X}).$$

Theorem 3.1. *Let $(Y_s)_{s>0}$ be a family of open sets in Y^R and let $\mathcal{N} \in Ob(D^b_{rh}(\mathcal{D}_Y))$. We make the assumptions of Theorem 2.3 with* Supp (G) *replaced by* Supp $(\mathcal{N})$ *and $SS(G)$ by* Char $(\mathcal{N})$ (the characteristic variety of $\mathcal{N}$). *Then $\int_f \mathcal{N}$ and $\int_f^{pr} \mathcal{N}$ belong to $Ob(D^b_{rh}(\mathcal{D}_X))$ and moreover:*

$$\mathrm{Char}\left(\int_f \mathcal{N}\right) \subset \tilde{\omega}\rho^{-1}(\mathrm{Char}\, \mathcal{N}),$$
$$\mathrm{Char}\left(\int_f^{pr} \mathcal{N}\right) \subset \tilde{\omega}\rho^{-1}(\mathrm{Char}\, \mathcal{N}),$$
$$\boldsymbol{R}f_*(\mathcal{N} \otimes^L_{\mathcal{D}_Y} \mathcal{O}_Y) = \left(\int_f \mathcal{N}\right) \otimes^L_{\mathcal{D}_X} \mathcal{O}_X,$$
$$\boldsymbol{R}f_!(\mathcal{N} \otimes^L_{\mathcal{D}_Y} \mathcal{O}_Y) = \left(\int_f^{pr} \mathcal{N}\right) \otimes^L_{\mathcal{D}_X} \mathcal{O}_X.$$

This theorem follows easily from Theorem 2.3 and the results in [2].

Corollary 3.2. *Let $f: Y \to X$ be a holomorphic map, with* $\dim X = 1$ *and $\mathcal{N}$ a regular holonomic $\mathcal{D}_Y$-Module. Let $x \in X$ and K a compact subset of $f^{-1}(x)$. Then there exists open neighborhoods U of x, V of K, with $V \subset f^{-1}(U)$ such that denoting by $f_V: V \to U$ the restriction of f, $\int_{f_V} (\mathcal{N}|_V)$ and $\int_{f_V}^{pr} (\mathcal{N}|_V)$ belong to $Ob(D^b_{rh}(\mathcal{D}_U))$ and that the conclusions of Theorem 3.1 are satisfied with f replaced by f_V.*

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