99. On Hilbert Modular Forms. III

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The purpose of this note is to give a final result on a problem posed in [3], which is concerned with the structure of the ring of Hilbert modular forms with integral Fourier coefficients. Let K be a real quadratic field and denote by $A_Z(\Gamma_K)_k$ the Z-module of symmetric Hilbert modular forms of weight k with integral Fourier coefficients. We put

$$A_Z(\Gamma_K)=\bigoplus_{k\geq 0} A_Z(\Gamma_K)_{2k}, \qquad A_Z^a(\Gamma_K)=\bigoplus_{k\geq 0} A_Z(\Gamma_K)_k.$$

Then $A_Z(\Gamma_K)$ is a graded subring of $A_Z^a(\Gamma_K)$. In [3], the author showed that the ring $A_Z(\Gamma_{Q(\sqrt{2})})$ is generated by three forms V_2, V_4 and V_6 over Z and $A_Z(\Gamma_{Q(\sqrt{5})})$ is generated by four forms W_2, W_6, W_{10} and W_{12}, where the subscripts denote the weight, and these modular forms are explicitly expressed by Eisenstein series (cf. [3], [4]).

In [5], H. L. Resnikoff showed the existence of a symmetric Hilbert modular form of odd weight 15 for $Q(\sqrt{5})$ by using Igusa-Hammond's modular imbedding, and he gave a quadratic relation it satisfies. We can show that Resnikoff's method is applicable in the case $K=Q(\sqrt{2})$.

From now on, we restrict ourselves to the case $K=Q(\sqrt{2})$. In this case, every element $f(\tau)$ in $A_Z^a(\Gamma_K)$ has the following Fourier expansion.

$$\begin{aligned} f(\tau)= &\sum_{\substack{\nu\gg 0\\ \nu\equiv 0\ (\mathrm{mod}\ 1/2\sqrt{2})}} a_f(\nu)\exp\,[2\pi i tr(\nu\tau)] \\ &= a_f(0)+a_f((-1+\sqrt{2})/2\sqrt{2})x^{-1}q+a_f(1/2)q \\ &\quad +a_f((1+\sqrt{2})/2\sqrt{2})xq+a_f((-2+2\sqrt{2})/2\sqrt{2})x^{-2}q^2 \\ &\quad +a_f((-1+2\sqrt{2})/2\sqrt{2})x^{-1}q^2+a_f(1)q^2 \\ &\quad +a_f((1+2\sqrt{2})/2\sqrt{2})xq^2+a_f((2+2\sqrt{2})/2\sqrt{2})x^2q^2 \\ &\quad +\cdots, \end{aligned}$$

where $\tau=(z_1, z_2)\in\mathfrak{H}\times\mathfrak{H}$, $q=\exp\,[\pi i(z_1+z_2)]$, $x=\exp\,[\pi i(z_1-z_2)]$. We denote by $G_k(\tau)$ the normalized Eisenstein series for the Hilbert modular group $\Gamma_K=SL(2, \mathfrak{o}_K)$. We put

$$\begin{aligned} &H_2=G_2, \qquad H_4=2^{-6}\cdot 3^{-2}\cdot 11(G_2^2-G_4), \\ &H_6=-2^{-8}\cdot 3^{-3}\cdot 13^{-1}\cdot 5\cdot 7^2G_2^3+2^{-8}\cdot 3^{-2}\cdot 5^{-1}\cdot 13^{-1}\cdot 11\cdot 59G_2G_4 \\ &\qquad -2^{-7}\cdot 3^{-3}\cdot 5^{-1}\cdot 13^{-1}\cdot 19^2G_6. \end{aligned}$$

If we use the notation in [3], then

$$H_2=V_2, \quad H_4=V_4, \quad H_6=V_6-V_2V_4.$$

Therefore, H_2, H_4 and H_6 form a set of generators of $A_Z(\Gamma_K)$. The Fourier expansions of H_k $(k=2,4,6)$ are given as follows:

$$H_2(\tau)=1+2^4\cdot 3\{(x^{-1}+3+x)q+(7x^{-2}+8x^{-1}+15+8x+7x^2)q^2+\cdots\}.$$
$$H_4(\tau)=(x^{-1}-2+x)q+(-4x^{-2}-8x^{-1}+24-8x-4x^2)q^2+\cdots.$$
$$H_6(\tau)=q+(-2x^{-2}-16x^{-1}+12-16x-2x^2)q^2+\cdots.$$

Now our main theorem specialized to the case $K=Q(\sqrt{2})$ is stated as follows.

Theorem 1. (1) *There exists a modular form H_9 of weight 9 for $Q(\sqrt{2})$ with integral Fourier coefficients, whose Fourier expansion is given by*

$$H_9(\tau)=q-(96x^{-1}+336+96x)q^2+\cdots.$$

(2) *The square H_9^2 can be expressed as*

$$H_9^2=H_6(H_2^3H_6+2^2H_2^2H_4^2-2^5\cdot 3^2H_2H_4H_6-2^{10}H_4^3-2^6\cdot 3^3H_6^2).$$

(3) *The four elements H_2, H_4, H_6 and H_9 form a minimal set of generators over Z of $A_Z^\alpha(\Gamma_{Q(\sqrt{2})})$.*

Remark 1. In [2], F. Hirzebruch determined the structure of the ring $A_C^\alpha(\Gamma_{Q(\sqrt{2})})$ by studying the Hilbert modular surface. He gave the same formula as in (2) by a different method (cf. [2], p, 316, (22)).

Remark 2. In [1], H. Cohn computed the explicit form of a modular equation for Hilbert modular functions over $Q(\sqrt{2})$. The above modular forms H_2, H_4 and H_6 appear in his computation (cf. [1], p. 230, (2.5)).

Remark 3. The calculations needed in the proof of (2) were performed with the cooperation of Hokkaido University Computing Center. (The author used the system "REDUCE".)

The proof of the above theorem is based on the general theory of the modular imbedding and Igusa's expression of a Siegel cusp form $(\chi_{35})^2$, where χ_{35} is a Siegel cusp form of degree 2 and weight 35 (cf. [5]).

In the case $K=Q(\sqrt{5})$, the generators W_k (k=2, 6, 10, 12) of $A_Z(\Gamma_K)$ have the following Fourier expansions.

$$W_2(\tau)=1+2^3\cdot 3\cdot 5\{(x^{-1}+x)q+(x^{-4}+5x^{-2}+6+5x^2+x^4)q^2+\cdots\},$$
$$W_6(\tau)=(x^{-1}+x)q+(x^{-4}+20x^{-2}-90+20x^2+x^4)q^2+\cdots,$$
$$W_{10}(\tau)=(x^{-2}-2+x^2)q^2+(-2x^{-5}-18x^{-3}+20x^{-1}+20x-18x^3-2x^5)q^3\cdots,$$
$$W_{12}(\tau)=q^2+(x^{-5}-15x^{-3}-10x^{-1}-10x-15x^3+x^5)q^3+\cdots,$$

where

$$q=\exp[\pi i(z_1+z_2)],\qquad x=\exp[\pi i(z_1-z_2)/\sqrt{5}].$$

The main result in the case $K=Q(\sqrt{5})$ is as follows.

Theorem 2. (1) *There exists a modular form W_{15} of weight 15 for $Q(\sqrt{5})$ with integral Fourier coefficients, whose Fourier expansion is*

$$W_{15}(\tau)=q^2-(x^{-5}+275x^{-1}+275x+x^5)q^3+\cdots.$$

(2) *The square* W_{15}^2 *can be expressed by* W_2, W_6, W_{10} *and* W_{12} *as*

$$W_{15}^2 = 5^5 W_{10}^3 - 2\cdot 3^3 W_6^5 + 2\cdot 5^2 W_2 W_6^3 W_{10} + 2\cdot 5^3 W_2 W_6 W_{10} W_{12} + W_2^3 W_{12}^2.$$

(3) *The five elements* W_2, W_6, W_{10}, W_{12} *and* W_{15} *form a minimal set of generators of* $A_Z^{\alpha}(\Gamma_{Q(\sqrt{5})})$.

References

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