94. A Remark on Constructions of Certain Normed and Nonsingular Bilinear Maps

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The following are classical problems of real algebra:

(1) For what triples (a, b, c) does there exist a real-bilinear map $f: \boldsymbol{R}^a \times \boldsymbol{R}^b \rightarrow \boldsymbol{R}^c$ with $\|f(x, y)\| = \|x\| \cdot \|y\|$ (so-called normed bilinear map)? (Here for $z \in \boldsymbol{R}^d$, $\|z\| := z_1^2 + \cdots + z_d^2$.)

(2) For what triples (a, b, c) does there exist a real-bilinear map $f: \boldsymbol{R}^a \times \boldsymbol{R}^b \rightarrow \boldsymbol{R}^c$ with the property: $f(x, y) = 0$ implies $x = 0$ or $y = 0$ (so-called nonsingular bilinear map)?

The purpose of this note is to announce that for some special cases, i.e. $b = c = 2^m$ in problem (1) and $a = b = 2^m$ in problem (2) these problems can be looked at in a way involving notions of graph-theory. In particular in problem (1) in the special case of $b = c = 2^m$ normed bilinear maps with maximal a can be constructed by cocliques as in Theorem 1. It is hoped that the approach taken in this note may simplify the matrix calculations needed for (1) and may show a certain duality between problems (1) and (2).

Let $W = \boldsymbol{F}_2^m$ and $W^* = \mathrm{Hom}_{\boldsymbol{F}_2}(W, \boldsymbol{F}_2)$ be dual m-dimensional vector spaces over the field $\boldsymbol{F}_2$. Let $V = W \times W^*$ be a $2m$-dimensional vector space over $\boldsymbol{F}_2$. The elements $v \in V$ have a representation $v = (w, \lambda)$ with $w \in W$, $\lambda \in W^*$. One defines a quadratic form Q (of hyperbolic type) on V by setting $Q(v) = \lambda(w) \in \boldsymbol{F}_2$.

To each element $v \in V$ we associate a real $2^m \times 2^m$ matrix $M(v)$ in the following way: Rows and columns of $M(v)$ are indexed by some enumeration of the elements of W, and for $M(v) = (m_{x,y})_{x,y \in W}$ we have $m_{x,y} = 0$ for $y \neq x + w$ and $m_{x,x+w} = (-1)^{\lambda(x)}$. $M(v)$ can also be identified with a linear transformation of the real group ring $\boldsymbol{R}[W]$ of W. This gives $M(v): \boldsymbol{R}[W] \rightarrow \boldsymbol{R}[W]$ as $M(v)(e_x) = (-1)^{\lambda(x)} \cdot e_{x+w}$.

From this definition one finds the following formulas:

(1) $M(v)$ is symmetric $\Longleftrightarrow Q(v) = 0$

$M(v)$ is skew $\Longleftrightarrow Q(v) = 1$

$M(v_1)M(v_2) = M(v_2)M(v_1) \Longleftrightarrow Q(v_1 + v_2) + Q(v_1) + Q(v_2) = 0$

$M(v_1)M(v_2) = -M(v_2)M(v_1) \Longleftrightarrow Q(v_1 + v_2) + Q(v_1) + Q(v_2) = 1$

$M(0) =$ identity-matrix.

The relations (1) give motivation to introduce the following graphs:

$V^+(2m)$ has as vertices all elements of V and $\{v_1, v_2\}$, $v_1 \neq v_2$ is an edge iff $Q(v_1+v_2)=0$.

$\mathrm{Alt}^+(2m)$ has as vertices all elements a of V with $Q(a)=1$ and $\{a_1, a_2\}$, $a_1 \neq a_2$ is an edge iff $Q(a_1+a_2)=0$.

These graphs give an elementary but useful setting for looking at the above mentioned problems in certain special cases.

Recall that a *coclique* in a graph is a set of vertices which are mutually non-connected.

Theorem 1. *Let $C \subset \mathrm{Alt}^+(2m)$ be a coclique with $|C|=k$ elements, $C=\{a_1, \cdots, a_k\}$ say. Then the bilinear map $f: \boldsymbol{R}^{k+1} \times \boldsymbol{R}^{2^m} \to \boldsymbol{R}^{2^m}$ defined by $f(e_0, x)=M(0)x = x$, $f(e_i, x)=M(a_i)x$ for $i=1, \cdots, k$ is normed ($e_0, \cdots, e_k$ are the canonical basis of $\boldsymbol{R}^{k+1}$).*

Let A be the point-point 0–1 incidence matrix of the graph $\mathrm{Alt}^+(2m)$, i.e. rows and columns of A are indexed by points of $\mathrm{Alt}^+(2m)$, and for $A=(\alpha_{a,b})_{a,b \in \mathrm{Alt}^+(2m)}$ we have

$$\alpha_{a,b}=\begin{cases}1 & \{a,b\} \text{ edge} \\ 0 & \{a,b\} \text{ non-edge} \\ 0 & a=b\end{cases}$$

For $m>1$ it is well known that the symmetric matrix A has exactly three eigenvalues, $k>r>s$ say, cf. [3]. With an obvious extension of the above notation for group rings, let $\boldsymbol{R}[\mathrm{Alt}^+(2m)]$ have a canonical orthonormal basis e_a, $a \in \mathrm{Alt}^+(2m)$ and regard $A: \boldsymbol{R}[\mathrm{Alt}^+(2m)] \to \boldsymbol{R}[\mathrm{Alt}^+(2m)]$. Let $\boldsymbol{R}[\mathrm{Alt}^+(2m)]=E_0 \perp E_1 \perp E_2$ be the decomposition of $\boldsymbol{R}[\mathrm{Alt}^+(2m)]$ into the eigenspaces E_0, E_1, E_2 belonging to k, r, s respectively. Denote by $P_2(a)$ the projections of e_a into E_2.

Recall (cf. [1]) that a set $B \subset \mathrm{Alt}^+(2m)$ is a *distributed set* (with respect to the second eigenspace E_2) iff there exists a vector $e \in E_2$ with $\langle e, P_2(b)\rangle>0$ for $b \in B$ and $\langle e, P_2(a)\rangle<0$ for $a \in \mathrm{Alt}^+(2m)\backslash B$.

Theorem 2. *Let $B \subset \mathrm{Alt}^+(2m)$ be a distributed set* (with respect to the second eigenspace E_2), *$B=\{b_1, \cdots, b_n\}$ say. Then the bilinear map $f: \boldsymbol{R}^{2^m} \times \boldsymbol{R}^{2^m} \to \boldsymbol{R}^{n+1}$ defined by $f_0(x, y)=\langle x, M(0)y\rangle=\langle x, y\rangle$, $f_i(x, y)=\langle x, M(b_i)y\rangle$ for $i=1, \cdots, n$ is nonsingular.*

Theorem 1 is due to J. Radon (cf. [2]). The proof consists of transforming the defining property of normed bilinear map into a system of matrix equations and using formulas (1) to observe that the $\{M(0), M(a_1), \cdots, M(a_k)\}$ form a set of solutions of this system whenever $\{a_1, \cdots, a_k\}$ is a coclique.

The proof of Theorem 2 is also completely elementary and uses certain 4×4-minors of real skew-symmetric square matrices of size $2^m \times 2^m$. A detailed proof of Theorem 2 will be given later.

Recently I obtained an analogous theorem to Theorem 2 for the graphs $V^+(2m)$, and a related theorem for the complement of $\mathrm{Alt}^+(2m)$ in $V^+(2m)$.

References

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