91. *Uniqueness and Non-Uniqueness in the Cauchy Problem for a Class of Operators of Degenerate Type. II*

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In this note, we shall consider uniqueness and non-uniqueness of C^∞-solutions of the Cauchy problem for a class of partial differential operators whose characteristics degenerate infinitely on the initial surfaces. For weakly hyperbolic operators of this type, many authors studied on the well-posedness of the Cauchy problem. See [2], [3] and [4] for example. Our sufficient condition for uniqueness on the lower order terms corresponds to that of [2]. Theorem 2 shows that this condition is, in a sense, necessary for uniqueness. Note that we assume here only C^0-regularity on the coefficients of the lower order terms of operators (see (A.2)). Considering the sharp results of [3] and [4], we may conclude that uniqueness depends also on the regularity of the coefficients of operators.

§1. **Preliminaries.** In order to describe the degeneracy of characteristics on the initial surfaces, we prepare some results. The argument in this section is due to Tahara [2].

Let $\mu(t)$ be a function on $[0, T]$ satisfying

(1.1) $\mu(t)>0$ for $t>0$, $\mu(t)=O(t)$ as $t\longrightarrow+0$,

(1.2) $\mu(t)\in C^1([0, T])\cap C^\infty((0, T])$,

(1.3) $\mu(t)^k$, $\mu(t)^k\mu'(t)\in C^k([0, T])$ for any $k\in N$.

We define $\sigma(t)$ by $\sigma(t)=\exp\left(-\int_t^T \mu(s)^{-1}\,ds\right)$. Then we have

Lemma 1 (Tahara [2], Prop. 6.4). *The following conditions are equivalent to each other*:

(1.4) $\mu(t)=o(t)$ *as* $t\longrightarrow+0$,

(1.5) $\sigma(t)=O(t^m)$ *as* $t\longrightarrow+0$ *for any* $m\geqq 0$,

(1.6) $\mu(t)^m\sigma(t)\in C^\infty([0, T])$ *for any* $m\in Z$.

Note that (1.5) implies that $t=0$ is a zero of infinite order of the function $\sigma(t)$. In what follows we assume that $\mu(t)$ and $\sigma(t)$ satisfy above conditions. And we continue $\mu(t)$ and $\sigma(t)$ smoothly to $t<0$.

Example. The functions

$$\mu(t)=t^{k+1}/k,\quad t/(-\log t)^k,\quad t^{k+1}\exp(-t^{-k})/k,\quad (k>0)$$

correspond respectively to

$$\sigma(t)=\exp(-t^{-k}),\quad \exp\{-(-\log t)^{k+1}/(k+1)\},\quad \exp\{-\exp(t^{-k})\}.$$

§ 2. A sufficient condition for uniqueness. Let U be an open neighborhood of the origin in $\boldsymbol{R}^{d+1}$ and let $P=P(t, x; D_t, D_x)$ be a partial differential operator, defined in U, of the form:

$$P=D_t^m+\sum_{j+|\alpha|\leqq m, j<m} a_{j,\alpha}(t, x)D_t^j D_x^\alpha,$$

where $D_t=-i\partial_t=-i\partial/\partial t$, $D_x=-i(\partial_{x_1}, \cdots, \partial_{x_d})$.

We impose following conditions on P.

(A.1) The coefficients $a_{j,\alpha}$ $(j+|\alpha|=m)$ belong to $C^\infty(U)$.

(A.2) The coefficients $a_{j,\alpha}$ $(j+|\alpha|<m)$ belong to $C^0(U)$.

(A.3) The principal symbol P_m of P is factorized as

$$P_m(t, x; \tau, \xi)=\prod_{j=1}^{m}(\tau-\sigma(t)\lambda_j(x; \xi)),$$

where λ_j are C^∞-functions in $U\times(\boldsymbol{R}^d\setminus\{0\})$, independent of t and homogeneous of degree 1 with respect to ξ.

(A.4) $\lambda_i\neq\lambda_j$ in $U\times(\boldsymbol{R}^d\setminus\{0\})$ $(i\neq j)$.

(A.5) $\operatorname{Im}\lambda_j\equiv 0$ or $\neq 0$ in $U\times(\boldsymbol{R}^d\setminus\{0\})$.

(A.6) There exists a principal type partial differential operator P' such that

$$\mu(t)^m P(t, x; D_t, D_x)=P'(t, x; \mu(t)D_t, \mu(t)\sigma(t)D_x).$$

Then we have

Theorem 1. *Under assumptions* (A.1)–(A.6), *there exists an open neighborhood U' of the origin in $\boldsymbol{R}^{d+1}$ such that any $u\in C^\infty(U)$ satisfying*

$$Pu=0, \quad D_t^j u|_{t=0}=0 \quad (0\leqq j\leqq m-1)$$

vanishes in U'.

Remark 1. Consider the following operator in $\boldsymbol{R}^2$:

(2.1) $\quad P=\partial_t^2-\exp(-2t^{-k})\partial_x^2+at^{-m}\exp(-t^{-k})\partial_x \quad (a\in \boldsymbol{C}).$

Assumption (A.6) implies $m\leqq k+1$. On the other hand, Tarama [3] showed that the necessary and sufficient condition for the C^∞-well-posedness of the Cauchy problem for P is:

(2.2) $\quad m\leqq k+1 \quad$ if $\operatorname{Re} a\neq 0$,

(2.3) $\quad m\leqq 2k+1 \quad$ if $\operatorname{Re} a=0$.

Hence (A.6) seems to be too strong. Nevertheless the following theorem shows that (A.6) is the best because we assume (A.2).

§ 3. Necessary conditions for uniqueness. Let Q be the following operator in $\boldsymbol{R}^{d+1}$:

$$Q=\partial_t^p+\exp(-qt^{-k})A(t, x; D_x)-t^{-m}\exp\{-(q-r)t^{-k}\}B(t, x; D_x),$$

where A and B are partial differential operators of order q and $q-r$ respectively with C^∞-coefficients in U and $p\geqq q>r\geqq 1$. Then we have

Theorem 2. *Assume*

(A.7) $\quad m>pr(k+1)/q$.

We also assume that there exist $\xi^0\in\boldsymbol{R}^d\setminus\{0\}$ and a branch $C(\xi^0)$ of $(B_{q-r}(0, 0; \xi^0)-A_q(0, 0; \xi^0))^{1/p}$ satisfying

(A.8) $\operatorname{Re} C(\xi^0) > 0$,

(A.9) $\operatorname{Re}\left\{\left(\dfrac{A_q(0,0;\xi^0)}{B_{q-r}(0,0;\xi^0)-A_q(0,0;\xi^0)}+1-\dfrac{q}{r}\right)C(\xi^0)\right\}>0.$

Then there exist $T>0$, *an open neighborhood* ω *of* 0 *in* $\boldsymbol{R}^d$, $u \in C^\infty((-T,T)\times\omega)$ *and* $f \in C^\infty((-T,T);C^0(\omega))$ *such that*

$$Qu-fu=0, \qquad (0,0)\in \operatorname{supp} u \subset \{t \geqq 0\}.$$

Remark 2. Assumptions (A.8) and (A.9) are the same as those in finitely degenerate case. See Theorem 1 of Nakane [1]. For the operator (2.1), (A.8) and (A.9) imply $\operatorname{Im} a \neq 0$. Hence Tarama's result shows the above function f cannot be C^∞.

Finally we give another non-uniqueness result corresponding to condition (2.3). Let Q be the operator in $\boldsymbol{R}^2$:

$$Q=\partial_t^p+\exp(-qt^{-k})D_x^q-t^{-m}\exp(-(q-r)t^{-k})D_x^{q-r},$$

where $p\geqq q>r\geqq 1$. Then we have

Theorem 3. *Suppose*

(A.10) $m>pr(2k+1)/q$.

Then there exist C^∞*-functions* u *and* f *in* $\boldsymbol{R}^2$ *satisfying*

$$Qu-fu=0, \qquad (0,0)\in \operatorname{supp} u \subset \{t \geqq 0\}.$$

Remark 3. When $p=q=2$ and $r=1$, Q is just the operetor (2.1) with $a=i$. Considering (2.3), we conclude that (A.10) is the best.

Remark 4. In Theorems 2 and 3, we consider non-uniqueness in case $\sigma(t)=\exp(-t^{-k})$. Similar results also hold for other $\sigma(t)$ mentioned in §1. Those results corresponding to Theorem 3 show the necessity of the sufficient condition for the C^∞-well-posedness of the Cauchy problem obtained in [4]. Detailed results will be published elsewhere.

References

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