67. Note on the Wiener Compactification and the H^p-Space of Harmonic Functions

By Hiroshi TANAKA*) and Joel L. SCHIFF**)

(Communicated by Kôsaku YOSIDA, M. J. A., June 14, 1983)

Introduction. Let R be a hyperbolic Riemann surface. We denote by $HB(R)$ (resp. $HB'(R)$) the class of all bounded harmonic (resp. quasibounded harmonic) functions on R. For p $(1<p<\infty)$, we denote by $H^p(R)$ the class of all harmonic functions u on R such that $|u|^p$ has a harmonic majorant. Then Naim [1] obtained in terms of the Martin boundary that $HB(R)\subset H^p(R)\subset HB'(R)$ for all p. On the other hand the second author [3] proved in terms of Wiener boundary that $\dim HB(R)<\infty$ implies $HB(R)=H^p(R)$ for all p. In this note we shall prove that if any two classes of $HB(R)$, $H^p(R)$ and $HB'(R)$ coincide, then we necessarily have $\dim HB(R)<\infty$. Thus we obtain that $\dim HB(R)<\infty$ if and only if $HB(R)=H^p(R)$ for some p and hence for all p. The first author wishes to express his thanks to Prof. A. Yoshikawa for valuable discussions on L^p-spaces.

1. The H^p-space of harmonic functions. Let R be a hyperbolic Riemann surface and let z_0 be a fixed point in R once for all. Let $\{R_n\}_{n=1}^{\infty}$ be a regular exhaustion of R such that z_0 is contained in all R_n's. We denote by $\mu_n=\mu_{z_0}^{R_n}$ the harmonic measure on the boundary ∂R_n. Note that $\int_{\partial R_n} d\mu_n=1$ for all n.

Definition. A harmonic function u on R belongs to $H^p(R)$, $1\leqq p<\infty$, if and only if the p-mean values

$$\|u\|_{p,n}=\left(\int_{\partial R_n}|u|^p d\mu_n\right)^{1/p}$$

are uniformly bounded in n. Set $H^\infty(R)=HB(R)$, the space of all bounded harmonic functions on R.

Theorem 1 (Naim [1]). (i) *$u\in H^p(R)$, $1\leqq p<\infty$, if and only if $|u|^p$ has a harmonic majorant.*

(ii) $HB(R)\subset H^p(R)\subset HB'(R)$, $1<p<\infty$.

2. Lemma on L^p-space. Let X be a compact Hausdorff space and μ be a positive (Radon) measure on X. We denote by $S\mu$ the support of μ. We note that $x\in X$ belongs to $S\mu$ if and only if $\mu(V)>0$ for any open neighborhood V of x. We denote by $L^p=L^p(X,\mu)$ the

*) Department of Mathematics, Joetsu University of Education, Joetsu.
**) Department of Mathematics, University of Auckland, Auckland, New Zealand.

equivalence class of all real-valued p-integrable functions on X with $1 \leqq p < \infty$ and by $L^\infty = L^\infty(X, \mu)$ the equivalence class of all essentially bounded measurable functions on X. We note that $L^q \subset L^p$ for all p and q with $1 \leqq p < q \leqq \infty$.

Lemma. (i) *If $S\mu$ is a finite set, then $L^p = L^q$ for all p and q with $1 \leqq p < q \leqq \infty$.*

(ii) *If $S\mu$ is not a finite set, then*

$$L^\infty \subsetneqq \bigcap_{1<p<\infty} L^q \subsetneqq L^q \subsetneqq L^p \subsetneqq \bigcup_{1<p<\infty} L^p \subsetneqq L^1$$

with $1 < p < q < \infty$.

Proof. (i) Suppose $S\mu$ is a finite set. Then we have $L^1 = L^\infty$ and hence $L^p = L^q$ for all p and q.

(ii) Suppose $S\mu$ is not a finite set. Let p and q be any fixed so that $1 \leqq p < q \leqq \infty$. Since X is a normal space, we can find a family $\{B_n\}_{n=1}^\infty$ of disjoint Borel sets in X such that $\mu(B_n) > 0$ for all n.

(a) The case of $q < \infty$: In this case we may assume that $0 < \mu(B_n) \leqq 2^{-nq/(q-p)}$ $(n = 1, 2, \cdots)$. For each n we choose $a_n > 0$ so that $a_n^p \mu(B_n) = 1/2^n$. If we set $f = \sum_{n=1}^\infty a_n \chi_{B_n}$ (χ_{B_n} is the characteristic function of B_n), then f belongs to $L^p - L^q$.

(b) The case of $q = \infty$: In this case we may assume that $0 < \mu(B_n) \leqq 1/n!$ $(n = 1, 2, \cdots)$. If we set $f = \sum_{n=1}^\infty 2^n \chi_{B_n}$, then f does not belong to L^∞. However, since

$$\int f^p d\mu = \sum_{n=1}^\infty (2^n)^p \mu(B_n) \leqq \exp(2^p) < \infty \qquad \text{for all } p \ (1 \leqq p < \infty)$$

we see that $f \in L^p - L$.

By the aid of (a) and (b) we can show that the first four inclusion relations are strict. On the other hand it follows from (a) that, for each $k = 1, 2, \cdots$, there is a non-negative f_k in $L^1 - L^{1+1/k}$ with $\int f_k d\mu = 1/2^k$. If we set $f = \sum_{k=1}^\infty f_k$, then we see that f belongs to $L^1 - L^{1+1/k}$ for all k and hence $L^1 - \bigcup_{k=1}^\infty L^{1+1/k} \neq \phi$. Since $\bigcup_{1<p<\infty} L^p = \bigcup_{k=1}^\infty L^{1+1/k}$, we complete the proof.

3. Main result. We shall denote by R^W the Wiener compactification of R and by Δ^W the harmonic boundary of R^W (cf. [2]). Let μ_z be the harmonic measure on Δ^W with respect to $z \in R$ and R^W. Set $\mu = \mu_{z_0}$ in the following. We note that Δ^W is a compact Hausdorff space and μ is a positive measure on Δ^W with $\mu(\Delta^W) = 1$.

Definition. We define a linear operator T on $L^1(\Delta^W, \mu)$ by

$$(Tf)(z) = \int_{\Delta^W} f(\zeta) d\mu_z(\zeta) \qquad (z \in R).$$

Theorem 2 (cf. [3]). *T is a linear bijection of $L^1(\Delta^W, \mu)$ onto $HB'(R)$. Furthermore*

$$T(L^p(\Delta^W, \mu)) = H^p(R) \qquad (1 < p < \infty),$$
$$T(L^\infty(\Delta^W, \mu)) = HB(R).$$

Proof. It follows from Theorem 4D in [2] that T is a linear bijection. The second half of the theorem follows from Theorem 4 in [3].

We denote by $C^1(\Delta^W, \mu)$ the equivalence class of the family of all real-valued, integrable and continuous (in the extended sense) functions on Δ^W. For any p $(1<p\leqq\infty)$ we set

$$C^p(\Delta^W, \mu)=L^p(\Delta^W, \mu)\cap C^1(\Delta^W, \mu).$$

Theorem 3. $C^1(\Delta^W, \mu)=L^1(\Delta^W, \mu)$.

Proof. Let $f\in L^1(\Delta^W, \mu)$ be any fixed. Since $(Tf)(z)$ can be continuously extended over Δ^W (cf. Theorem 4D, IV, in [2]), we again denote by Tf the continuous extension of $(Tf)(z)$ over Δ^W. Furthermore we denote by Sf the restriction of Tf to Δ^W. Then, by definition, S is a linear bijection of $L^1(\Delta^W, \mu)$ onto $C^1(\Delta^W, \mu)$. This completes the proof.

Corollary. *$C^p(\Delta^W, \mu)=L^p(\Delta^W, \mu)$ for all p $(1\leqq p\leqq\infty)$.*

We shall denote by O^n_{HB} the class of all hyperbolic Riemann surfaces such that $HB(R)$ has at most dimension n $(n=1, 2, \cdots)$. Then it is known (cf. [2]) that R belongs to O^n_{HB} if and only if Δ^W consists of at most n points.

Theorem 4. *All of the following inclusion relations are identical or strict accordingly as R belongs to $\bigcup_{n=1}^{\infty} O^n_{HB}$ or not:*

$$HB(R)\subset \bigcap_{1<q<\infty} H^q(R)\subset H^q(R)\subset H^p(R)\subset \bigcup_{1<p<\infty} H^p(R)\subset HB'(R)$$
$$(1<p<q<\infty).$$

Proof. Since $S\mu$ equals Δ^W (cf. [2]), the theorem is an immediate consequence of Theorem 2 and Lemma.

Remark. (i) If R is a hyperbolic plane domain, then $\dim HB(R)=\infty$. Thus the six classes in Theorem 3 are distinct.

(ii) The second author [3] proved that $\dim HB(R)<\infty$ implies $HB(R)=HB'(R)$ and hence $H^p(R)=HB(R)$ for all p $(1<p<\infty)$.

References

[1] Lumer-Naim, L.: H^p-spaces of harmonic functions. Ann. Inst. Fourier, **17**, 425–469 (1967).

[2] Sario, L., and M. Nakai: Classification Theory of Riemann Surfaces. Berlin-Heidelberg-New York, Springer (1970).

[3] Schiff, J. L.: H^p-spaces of harmonic functions and the Wiener compactification. Math. Z., **132**, 135–140 (1973).