64. On the Isomonodromic Deformation of a Linear Ordinary Differential Equation of the Third Order

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§ 1. Introduction. Consider a third order linear ordinary differential equation of Fuchsian type

$$\frac{d^3y}{dx^3}+p_1\frac{d^2y}{dx^2}+p_2\frac{dy}{dx}+p_3y=0 \tag{1.1}$$

with the following Riemannian scheme:

$$\left\{\begin{matrix} x=0 & x=1 & x=t & x=\lambda_j & x=\infty \\ \alpha_0 & \alpha_1 & \beta & \gamma_j & \alpha_\infty \\ \alpha_0+\kappa_0 & \alpha_1+\kappa_1 & \beta+\theta & \gamma_j+2 & \alpha_\infty+\kappa_\infty \\ \alpha_0+\kappa_0' & \alpha_1+\kappa_1' & \beta+\theta' & \gamma_j+3 & \alpha_\infty+\kappa_\infty' \end{matrix}\right\} \tag{1.2}$$

$$(j=1,2,3,4)$$

and we suppose that *the singularities* $x=\lambda_j$ $(j=1,2,3,4)$ *are non-logarithmic ones and the characteristic exponents at each singular point do not differ by integer.*

The purpose of this paper is to derive a system of isomonodromic deformation equations of (1.1) regarding t as deformation parameter.

§ 2. Hamiltonian system attached to (1.1). The coefficients $p_j(x)$ $(j=1,2,3)$ of the equation (1.1) are given by

$$p_1(x)=\frac{a_0^1}{x}+\frac{a_1^1}{x-1}+\frac{b^1}{x-t}+\sum_{k=1}^{4}\frac{c_k^1}{x-\lambda_k},$$

$$p_2(x)=\frac{a_0^2}{x^2}+\frac{a_1^2}{(x-1)^2}+\frac{b^2}{(x-t)^2}+\sum_{k=1}^{4}\frac{c_k^2}{(x-\lambda_k)^2}$$
$$+\frac{a_\infty^2}{x(x-1)}-\frac{t(t-1)H}{x(x-1)(x-t)}+\sum_{k=1}^{4}\frac{\lambda_k(\lambda_k-1)\mu_k}{x(x-1)(x-\lambda_k)},$$

$$p_3(x)=\frac{a_0^3}{x^3}+\frac{a_1^3}{(x-1)^3}+\frac{b^3}{(x-t)^3}+\sum_{k=1}^{4}\frac{c_k^3}{(x-\lambda_k)^3}$$
$$+\frac{1}{T(x)}\Big[a_\infty^3+\eta_0\frac{t}{x}-\eta_1\frac{t-1}{x-1}+\eta_t\frac{t(t-1)}{x-t}$$
$$+\sum_{k=1}^{4}\Big\{\frac{T(\lambda_k)}{(x-\lambda_k)^2}+\frac{T'(\lambda_k)}{x-\lambda_k}\Big\}\xi_k+\sum_{k=1}^{4}\frac{\zeta_k}{x-\lambda_k}\Big],$$

where

$$T(x)=x(x-1)(x-t)$$

and a_Δ^i, b^i, c_k^i $(i=1,2,3;\ k=1,2,3,4;\ \Delta=0,1,\infty)$ are constants determined by the characteristic exponents.

We see from the assumption that $x=\lambda_k$ are non-logarithmic singularities that η_Δ, ξ_j, ζ_j ($\Delta=0, 1, t$; $j=1, 2, 3, 4$) and H are determined as rational functions of t, λ_k, μ_k ($k=1, 2, 3, 4$). Using H thus determined, we obtain the following theorem.

Theorem 1. *The isomonodromic deformation of* (1.1) *is governed by the Hamiltonian system*

$$H(\alpha_0, \alpha_1, \beta, \gamma_k) \qquad \begin{cases} \dfrac{d\lambda_j}{dt}=\dfrac{\partial H}{\partial \mu_j} \\ \dfrac{d\mu_j}{dt}=-\dfrac{\partial H}{\partial \lambda_j} \end{cases} \qquad (j=1, 2, 3, 4)$$

with the Hamiltonian

$$H=-\operatorname*{Res}_{x=t} p_2(x)$$

and the canonical variables conjugate to λ_j:

$$\mu_j=\operatorname*{Res}_{x=\lambda_j} p_2(x) \qquad (j=1, 2, 3, 4).$$

The explicit form of Hamiltonian function H for the system $H(0, 0, 0, 0)$ is given in the last section.

§ 3. Canonical transformation. Consider a transformation of (1.1) of the form

$$y=\Phi(x)z. \tag{3.1}$$

Putting in (3.1)

$$\Phi(x)=x^{\alpha_0}(x-1)^{\alpha_1}(x-t)^{\beta}\prod_{k=1}^{4}(x-\lambda_k)^{\gamma_k},$$

we obtain by this transformation the linear equation

$$\frac{d^3z}{dx^3}+q_1\frac{d^2z}{dx^2}+q_2\frac{dz}{dx}+q_3z=0 \tag{3.2}$$

having the Riemannian scheme (1.2) with

$$\alpha_0=\alpha_1=\beta=\gamma_k=0 \qquad (k=1, 2, 3, 4).$$

A simple computation shows that the coefficients $q_i(x)$ ($i=1, 2, 3$) of (3.2) are related to those of (1.1) as

$$\begin{aligned} q_1(x)&=p_1(x)+3\Phi'\Phi^{-1}, \\ q_2(x)&=p_2(x)+[2p_1(x)\Phi'+3\Phi'']\Phi^{-1}, \\ q_3(x)&=p_3(x)+[p_2(x)\Phi'+p_1(x)\Phi''+\Phi''']\Phi^{-1}. \end{aligned} \tag{3.3}$$

Set

$$K=-\operatorname*{Res}_{x=t} q_2(x), \qquad \nu_k=\operatorname*{Res}_{x=\lambda_k} q_2(x),$$

then the relation (3.3) reads

$$K=H-\operatorname*{Res}_{x=t}(2p_1(x)\Phi'+3\Phi'')\Phi^{-1}, \tag{3.4}$$

$$\nu_k=\mu_k+\operatorname*{Res}_{x=\lambda_k}(2p_1(x)\Phi'+3\Phi'')\Phi^{-1}. \tag{3.5}$$

Then we can prove

Theorem 2. *The change of variables* (3.4), (3.5):

$$(\lambda_k, \mu_k, H) \longrightarrow (\lambda_k, \nu_k, K)$$

defines a canonical transformation, which takes the Hamiltonian system $H(\alpha_0, \alpha_1, \beta, \gamma_k)$ to $H(0, 0, 0, 0)$.

Remark 1. The transformation in the above theorem is invertible. Hence the Hamiltonian systems $H(\alpha_0, \alpha_1, \beta, \gamma_k)$ are transformed to each other by the canonical transformation.

Remark 2. The transformation (3.1) with

$$\Phi(x) = \exp\left(-\frac{1}{3}\int^x p_1(x)dx\right)$$

takes (1.1) into the linear equation (3.2) with $q_1(x) \equiv 0$. The linear equation of this form is called *of SL-type*.

§ 4. Hamiltonian. We will give the explicit form of the Hamiltonian function for the system $H(0, 0, 0, 0)$. Suppose that $\alpha_0 = \alpha_1 = \beta = \gamma_k = 0$ for the equation (1.1). Then the condition that $x = \lambda_j$ $(j=1, 2, 3, 4)$ are non-logarithmic singular points reads as

$$\xi_i = -2(\mu_j + E_j) \tag{4.1}$$

$$\zeta_j = \frac{1}{2}\xi_j(\mu_j + \xi_j), \qquad (j=1, 2, 3, 4) \tag{4.2}$$

$$\xi_j(\xi_j + F_j) + 2G_j = 0, \tag{4.3}$$

where E_j, F_j and G_j are constant terms in the Laurent series expansion of $p_1(x)$, $p_2(x)$ and $p_3(x)$ at $x = \lambda_j$ respectively.

Solving (4.3) with respect to H, we arrive at the

Proposition. *The Hamiltonian function H for the system $H(0, 0, 0, 0)$ is given by*

$$\left[t(t-1)\sum_{k=1}^{4}(\mu_k + E_k)\frac{T(\lambda_k)}{\Lambda'(\lambda_k)}\right]H = \sum_{j=1}^{4} A_j \frac{T(\lambda_j)}{\Lambda'(\lambda_j)},$$

where

$$\begin{aligned} A_j = (\mu_j + E_j)T(\lambda_j)\Big[&(\mu_j + 2E_j)\left(\mu_j + E_j - \frac{T'(\lambda_j)}{T^2(\lambda_j)}\right) \\ &-\left(\frac{1}{\lambda_j} + \frac{1}{\lambda_j - 1}\right)\mu_j + \sum_{\substack{k=1\\k\neq j}}^{4}\frac{\lambda_k(\lambda_k - 1)\mu_k}{\lambda_j(\lambda_j - 1)(\lambda_j - \lambda_k)} + U_j\Big] \\ &+\sum_{\substack{k=1\\k\neq j}}^{4}\frac{\mu_k + E_k}{\lambda_j - \lambda_k}\left[\mu_k + 2E_k - 2T'(\lambda_k) - \frac{2T(\lambda_k)}{\lambda_j - \lambda_k}\right] + a_\infty^3, \end{aligned}$$

$$U_j = \frac{a_0^2}{\lambda_j^2} + \frac{a_1^2}{(\lambda_j - 1)^2} + \frac{b^2}{(\lambda_j - t)^2} + \sum_{\substack{k=1\\k\neq j}}^{4}\frac{2}{(\lambda_j - \lambda_k)^2} + \frac{a_\infty^2}{\lambda_j(\lambda_j - 1)} + \frac{T'(\lambda_j)}{T(\lambda_j)}$$

and

$$\Lambda(x) = \prod_{j=1}^{4}(x - \lambda_j).$$

Remark 3. ξ_j, ζ_j $(j=1, 2, 3, 4)$ are determined by (4.1), (4.2) as rational functions of λ_k, μ_k $(k=1, 2, 3, 4)$ and η_Δ $(\Delta = 0, 1, t)$ by (4.3).

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