45. ∇-Poincaré's Lemma and ∇-de Rham Cohomology for an Integrable Connection with Irregular Singular Points

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Let M be a complex manifold and let H be a divisor on M. Denote by Ω^p the sheaf over M of germs of holomorphic p-forms and denote by $\Omega^p(*H)$ the sheaf over M of germs of meromorphic p-forms which are holomorphic in $M-H$ and have poles on H for $p=0, \cdots, n$. In case $p=0$, we use frequently $\mathcal{O}$ and $\mathcal{O}(*H)$ instead of Ω^0 and $\Omega^0(*H)$, respectively.

We suppose throughout this paper that the divisor H has at most normal crossings.

Let $\mathcal{S}$ be a locally free sheaf of $\mathcal{O}$-modules of rank m on M. For each point x in M, there exists a neighborhood U over which $S|_U$ is isomorphic to $(\mathcal{O}|_U)^m=\mathcal{O}\otimes_C C^m$. Denote the isomorphism by g_U. Define the locally free sheaf $\mathcal{S}\Omega^p(*H)$ of $\mathcal{O}(*H)$-modules of rank m over M by

$$\mathcal{S}\Omega^p(*H)=\mathcal{S}\otimes_{\mathcal{O}}\Omega^p(*H),$$

for $p=0, \cdots, n$. For $p=0$, instead of $\mathcal{S}\Omega^0(*H)$, we use frequently $\mathcal{S}(*H)$ of which the restriction to U, $\mathcal{S}(*H)|_U$ is isomorphic to $(\mathcal{O}(*H))^m|_U =(\mathcal{O}^m\otimes\mathcal{O}(*H))|_U$ by the mapping $g_U\otimes id$, which will be denoted also by g_U.

Let ∇ be a connection on $\mathcal{S}(*H)$: ∇ is an additive mapping of $\mathcal{S}(*H)$ into $\mathcal{S}(*H)\otimes_{\mathcal{O}(H*)}\Omega^1(*H)=\mathcal{S}(*H)\otimes_{\mathcal{O}}\Omega^1=\mathcal{S}\otimes_{\mathcal{O}}\Omega^1(*H)=\mathcal{S}\Omega^1(*H)$ satisfying "Leibnitz rule"

$$\nabla(f\cdot u)=u\otimes df+f\cdot\nabla(u)$$

for all sections $f\in\mathcal{O}(*H)(U)$, $u\in\mathcal{S}\Omega^1(*H)(U)$. We suppose that the connection is integrable, that is, the composite mapping

$$\nabla^2 : \mathcal{S}(*H)\longrightarrow\mathcal{S}\Omega^1(*H)\longrightarrow\mathcal{S}\Omega^2(*H)$$

is a zero mapping.

If we take adequately an open covering $\{U_k\}$ on M, then to give connection ∇ means the following; for each U_k, the mapping

$$g_{U_k}\circ\nabla\circ g_{U_k}^{-1} : (\mathcal{O}(*H)|_{U_k})^m\longrightarrow(\mathcal{O}(*H))^m\otimes_{\mathcal{O}}\Omega^1|_{U_k},$$

is induced by a mapping

$$\nabla_k : (\mathcal{O}(*H)(U_k))^m\longrightarrow((\mathcal{O}(*H)\otimes_{\mathcal{O}}\Omega^1)(U_k))^m,$$

which is represented by $(d+\Omega_k)$ under a generator system $\langle e_{k,1}, \cdots, e_{k,m}\rangle$

of $(\mathcal{O}(U_k))^m$ (not $(\mathcal{O}(*H)(U_k))^m$), i.e.

$$\nabla_k(\langle e_{k,1}, \cdots, e_{k,m}\rangle u)=\langle e_{k,1}, \cdots, e_{k,m}\rangle(du+\Omega_k u)$$

where Ω_k is an m by m matrix of meromorphic 1-forms on U_k at most with poles on $U_k \cap H$; let $x_1, \cdots, x_n$ be holomorphic local coordinates on U_k and suppose $U_k \cap H=\{x_1 \cdots x_{n''}=0\}$, then Ω_k is of the form

$$\Omega_k=\sum_{i=1}^{n''} x^{-p_i}x_i^{-1}A_i(x)dx_i+\sum_{i=n''+1}^{n} x^{-p_i}A_i(x)dx_i,$$

where $p_i=(p_{i1}, \cdots, p_{in''}, 0, \cdots, 0) \in N^n$ and $A_i(x)$ is an m-by-m matrix of holomorphic functions in U_k for $i=1, \cdots, n$. The connection ∇ is integrable if and only if $d\Omega_k+\Omega_k \wedge \Omega_k=0$ for any k. For any k, k', denote by $g_{kk'}$ the isomorphism

$$g_{kk'}: (\mathcal{O}(U_k \cap U_{k'}))^m \longrightarrow (\mathcal{O}(U_k \cap U_{k'}))^m$$

induced by the isomorphism

$$g_{U_k} \circ g_{U_{k'}}^{-1}: (\mathcal{O}|_{U_k \cap U_{k'}})^m \longrightarrow (\mathcal{O}|_{U_k \cap U_{k'}})^m.$$

Then, by using the generator systems, $g_{kk'}$ is represented by $G_{kk'}$ a matrix of elements in $\mathcal{O}(U_k \cap U_{k'})$, i.e.

$$g_{kk'}\langle e_{k',1}, \cdots, e_{k',m}\rangle=\langle e_{k',1}, \cdots, e_{k',m}\rangle G_{kk'},$$

and

$$\Omega_{k'}=G_{kk'}^{-1}\,dG_{kk'}+G_{kk'}^{-1}\,\Omega_k\,G_{kk'}$$

in $U_k \cap U_{k'}$.

Denote by M^- the real blow up along the normal crossing divisor H and denote by pr the natural projection from M^- into M; for the real blow up and the notation used in the following, we refer to the preceding article [8]. For the sheaf $E=\mathcal{A}^-$, $\mathcal{A}'^-$, $\mathcal{A}_0^-$, $\mathcal{A}^-(*H)$ and $\mathcal{A}'^-(*H)$ over M^-, denote by $\mathcal{S}^-$, $\mathcal{S}'^-$, $\mathcal{S}_0^-$, $\mathcal{S}^-(*H)$ and $\mathcal{S}'^-(*H)$ the locally free sheaf $E\otimes_{pr^*\mathcal{O}}pr^*\mathcal{S}$ of E-modules over M^-, respectively. Moreover, denote by $\mathcal{S}^-\Omega^p(*H)$, $\mathcal{S}'^-\Omega^p(*H)$ and $\mathcal{S}_0^-\Omega^p$ the locally free sheaves $pr^*\Omega^p(*H)\otimes_{pr^*\mathcal{O}}\mathcal{S}^-$, $pr^*\Omega^p(*H)\otimes_{pr^*\mathcal{O}}\mathcal{S}'^-$ and $pr^*\Omega^p\otimes_{pr^*\mathcal{O}}\mathcal{S}_0^-$ of $\mathcal{A}^-(*H)$, $\mathcal{A}'^-(*H)$ and $\mathcal{A}_0^-$-modules over M^- for $p=1, \cdots, n$, respectively. Then, in a natural way, we can define the connections

$$\nabla^-: S^-(*H) \longrightarrow S^-\Omega^1(*H),$$
$$\nabla'^-: S'^-(*H) \longrightarrow S'^-\Omega^1(*H),$$
$$\nabla_0^-: S_0^- \longrightarrow S_0^-\Omega^1.$$

By the integrability, we can consider the complexes of sheaves

$$S^-(*H) \xrightarrow{\nabla} S^-\Omega^1(*H) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} S^-\Omega^n(*H) \longrightarrow 0$$
$$S'^-(*H) \xrightarrow{\nabla} S'^-\Omega^1(*H) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} S'^-\Omega^n(*H) \longrightarrow 0$$
$$S_0^- \xrightarrow{\nabla} S_0^-\Omega^1 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} S_0^-\Omega^n \longrightarrow 0,$$

where we write ∇ for ∇^-, ∇'^-, ∇_0^-.

Suppose here that ∇ satisfies the following condition: For any point $p \in H$, under the local representation of ∇,

(H.1) $p_i=0$ and $A_i(0)$ has no eigenvalue of integer for all $i \in [1, n]$ or

(H.2) $p_{ii}>0$ and $A_i(0)$ is invertible or $p_i=0$ and $A_i(0)$ has no eigenvalue of integer for all $i\in[1, n'']$.

Then, we can assert

Theorem 1. *If the assumption* (H.1) *is satisfied for any point in* H, *then the above three sequences are exact. If* (H.1) *or* (H.2) *is satisfied for any point on* H, *then the above sequences are exact except the second.*

Remark 1. Theorem 1 implies that locally the completely integrable system of partial differential equations of the first order

$$(e_i\partial/\partial x_i)u=x^{-p_i}A_i(x)u+x^{-q_i}b_i(x), \qquad i=1, \cdots, n',$$

can be solved in the category of functions strongly asymptotically developable under (H.1) or (H.2), where n' and n'' are positive integer equal or inferior to n, $e_i=x_i$ $(i=1, \cdots, n'')$, $e_i=1$ $(n''<i\leqq n)$, $p_i=(p_{i1}, \cdots, p_{in''}, 0, \cdots, 0)\in \boldsymbol{N}^n$, $(q_{i1}, \cdots, q_{in''}, 0, \cdots, 0)\in \boldsymbol{N}^n$, $A_i(x)$ is an m-by-m matrix of holomorphic functions at the origin in $\boldsymbol{C}^n$ and $b_i(x)$ is an m-vector of functions holomorphic and strongly asymptotically developable in an open polysector S at the origin in $\boldsymbol{C}^n$, for $i=1, \cdots, n$.

Moreover, we consider the complex of global section level:

$$\mathrm{GSK}^{\cdot}: \mathcal{S}(*H)(M)^m \xrightarrow{\nabla} \mathcal{S}\Omega^1(*H)(M)^m \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{S}\Omega^n(*H)(M)^m \longrightarrow 0.$$

Then, we can prove

Theorem 2. *If* $H^1(M, \mathcal{S})=0$ *and if* (H.1) *or* (H.2) *is satisfied for any point on* H, *then the following isomorphim is valid:*

$$H^1(\mathrm{GSK}^{\cdot})=H^1(M^-, \mathrm{Ker}\,\nabla_0^-),$$

here $\mathrm{Ker}\,\nabla_0^-$ *denotes the sheaf of solutions of* ∇_0^-.

Remark 2. From the short exact sequences

$$0 \longrightarrow \mathcal{S}_0^- \longrightarrow \mathcal{S}'^- \longrightarrow pr^*(\mathcal{O}_{\widehat{M|H}}\otimes_{\mathcal{O}}\mathcal{S}) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{S}_0^- \longrightarrow \mathcal{S}'^-(*H) \longrightarrow pr^*(\mathcal{O}_{\widehat{M|H}}(*H)\otimes_{\mathcal{O}}\mathcal{S}) \longrightarrow 0,$$

we can deduce the long exact sequences, and we see that the images of the mappings from $H^1(M^-, \mathcal{S}_0^-)$ to $H^1(M^-, \mathcal{S}'^-)$ and $H^1(M^-, \mathcal{S}'^-(*H))$ are zero if $H^1(M, \mathcal{S})=0$. This fact is the key to the proof of Theorem 2.

Remark 3. Let S be a locally free sheaf of $\mathcal{O}(*H)$-modules on M and let $\nabla: \mathcal{S}\to\mathcal{S}\otimes_{\mathcal{O}(*H)}\Omega^1(*H)$ be a integrable connection. Then, Theorems 1 and 2 are valid for this connection under the conditions (H.1) or (H.2).

The detail will be published elsewhere (see Majima [9]).

Finally, we give a conjecture (cf. Majima [6]).

Conjecture. Consider an integrable connection

$$\nabla: \mathcal{S} \longrightarrow \mathcal{S}\otimes_{\mathcal{O}(*H)}\Omega^1(*H),$$

where S is a locally free sheaf of $\mathcal{O}(*H)$-modules on M. Then, there exists a complex manifold X constructed by complex blow-ups etc., with the natural projection $h: X \to M$, such that

$$H^1(\mathrm{GSK}^{\cdot}) = H^1(X^-, \mathrm{Ker}\, \nabla_0^{X^-}),$$

where X^- is the real blow up along $h^{-1}(H)$ and $\nabla_0^{X^-}$ is defined as above for X^- instead of M^-.

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