40. *Ergodic Theorems for Semigroups of Operators on a Grothendieck Space*

By Sen-Yen SHAW
National Central University, Chung-Li, Taiwan

(Communicated by Kôsaku YOSIDA, M. J. A., April 12, 1983)

1. Introduction. A theorem of Atalla [1] states that the Cesàro means $\{(1/n)(T+\cdots+T^n)\}$ of a linear contraction T on a Grothendieck space X converge strongly if and only if the weak* closure and the strong closure of the range of T^*-I^* (the dual operator of $T-I$) coincide. The main purpose of the present paper is to prove an analogue of this theorem for semigroups of linear operators.

Throughout the paper X is a Grothendieck space, i.e., weak* sequential convergence in the dual space X^* is equivalent to weak sequential convergence (cf. [3]), and $\{T(t)\}_{t>0}$ is a locally integrable semigroup of linear operators on X. By this we mean that for each $x\in X$ $T(\cdot)x$ is strongly measurable on $(0, \infty)$ and $\int_0^t \|T(s)x\|\,ds<\infty$ for every $t\in(0, \infty)$. Then the Bochner integral $\int_0^t T(s)xds$ exists for all $x\in X$. Since $T(\cdot)$ is strongly continuous on $(0, \infty)$ (see [4, p. 616]), this integral is also an improper Riemann integral.

Let $S(t)$ denote the operator on X such that $S(t)x=\int_0^t T(s)\,xds$ for all $x\in X$. Then $S(t)$ is a continuous linear operator (see [4, p. 685]). The ergodic theory is concerned with the existence of $\lim_{t\to\infty} t^{-1}S(t)x$. When the limit exists strongly for all x in X, $T(\cdot)$ is said to be strongly ergodic.

First we specify some notations. P will stand for the map which sends x to the strong limit s-$\lim_{t\to\infty} t^{-1}S(t)x$ whenever the limit exists; its domain $D(P)$ is the set of all x for which the limit exists. Similarly, Q is the map in X^* determined by the weak* limits w*-$\lim_{t\to\infty} t^{-1}S^*(t)x^*$. Also we shall use the following notations:

$$\boldsymbol{F}=\bigcap_{t>0} N(T(t)-I); \qquad \boldsymbol{F}^*=\bigcap_{t>0} N(T^*(t)-I^*);$$

$$\boldsymbol{R}=\text{span}\left\{\bigcup_{t>0} R(T(t)-I)\right\}; \qquad \boldsymbol{R}^*=\text{span}\left\{\bigcup_{t>0} R(T^*(t)-I^*)\right\},$$

where $N(L)$ and $R(L)$ are the null space and the range of an operator L.

We shall prove the following theorems.

Theorem 1. *Let $T(\cdot)$ be a locally integrable semigroup of operators on a Grothendieck space X. Assume that*

(a) $\overline{\lim}_{t\to\infty} t^{-1}\|S(t)\|<\infty$ *and*

(b) $\text{s-lim}_{t\to\infty} t^{-1}T(t)S(u)x=0$ *for all* $x\in X$ *and* $u>0$.

Then the following two statements hold:

(i) *P is a bounded linear projection in X with $R(P)=\boldsymbol{F}$, $N(P)$ $=$s-closure $(\boldsymbol{R})$ and $D(P)=\{x\in X;\ \exists t_n\to\infty \ni \text{w-lim}_{n\to\infty} t_n^{-1}S(t_n)x$ exists$\}$.*

(ii) *Q is a bounded linear projection in X^* with $R(Q)=\boldsymbol{F}^*$, $N(Q)$ $=$s-closuse $(\boldsymbol{R}^*)$ and $D(Q)=\{x^*\in X^*;\ \exists t_n\to\infty \ni \text{w}^*\text{-lim}_{n\to\infty} t_n^{-1}S^*(t_n)x^*$ exists$\}$.*

When $T(\cdot)$ is a (C_0)-semigroup with generator A, the sets $\boldsymbol{F}$, $\boldsymbol{F}^*$, $\boldsymbol{R}$ and $\boldsymbol{R}^*$ can be replaced by $N(A)$, $N(A^*)$, $R(A)$ and $R(A^*)$, respectively.

Theorem 2. *A locally integrable semigroup $T(\cdot)$ of operators on a Grothendieck space is strongly ergodic if and only if the conditions* (a) *and* (b) *hold and so does the condition:*

(c) *w^*-closure $(\boldsymbol{R}^*)=$s-closure $(\boldsymbol{R}^*)$.*

Moreover, if $T(\cdot)$ is a (C_0)-semigroup with the infinitesimal generator A, the assertion holds with (c) *replaced by*

(c′) *w^*-closure $(R(A^*))=$s-closure $(R(A^*))$.*

Because the weak topology and the weak* topology in a reflexive space are identical, and because every strongly closed convex set is weakly closed; the above theorem immediately yields the following well-known

Corollary 3 (Masani [5]). *Let $T(\cdot)$ be a locally integrable semigroup on a reflexive Banach space. Then $T(\cdot)$ is strongly ergodic if and only if the conditions* (a), (b) *hold.*

2. Proofs of the theorems. Lemma 1 ([5, Lemma 2.3]). *The following identities hold:*

$$\begin{aligned} S(u)(T(t)-I)&=(T(u)-I)S(t)\\ &=S(t)(T(u)-I)\\ &=S(t+u)-S(t)-S(u) \qquad (u,\ t>0). \end{aligned}$$

Proof of Theorem 1. The assertion (i) has been proved in Shaw [6] for a general Banach space X. Here we shall prove (ii). First, Q is a bounded operator because of (a) and the estimation:

$$\begin{aligned} |\langle x, Qx^*\rangle| &=\underline{\lim}\,|\langle x, t^{-1}S^*(t)x^*\rangle|=\underline{\lim}\,|\langle t^{-1}S(t)x, x^*\rangle|\\ &\le\underline{\lim}\, t^{-1}\|S(t)\|\|x\|\|x^*\| \qquad (x\in X,\ x^*\in X^*). \end{aligned}$$

Next, we show that $D(Q)$ is strongly closed. Let $\{x_n^*\}\subset D(Q)$ and $x_n^*\to x^*$. Then $\{Qx_n^*\}$ is a Cauchy sequence with the limit $y^*\in X^*$. Fix an arbitrary $x\in X$ and then choose an n such that

$$\|x_n^*-x^*\|<\Big(\|x\|\sup_{t>1}\|S(t)\|/t\Big)^{-1}(\varepsilon/3) \quad \text{and} \quad \|Qx_n^*-y^*\|<(\varepsilon/3)\|x\|.$$

$x_n^*\in D(Q)$ implies the existence of a t_0 such that $|\langle x, t^{-1}S^*(t)x_n^*-Qx_n^*\rangle|$ $<\varepsilon/3$ for all $t>t_0$. Thus we have for $t>t_0$

$$\begin{aligned} |\langle x, t^{-1}S^*(t)x^*-y^*\rangle| &\le|\langle x, t^{-1}S^*(t)(x^*-x_n^*)\rangle|\\ &\quad+|\langle x, t^{-1}S^*(t)x_n^*-Qx_n^*\rangle|+|\langle x, Qx_n^*-y^*\rangle| \end{aligned}$$

$$<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon.$$

Hence $x^* \in D(Q)$ and $y^*=\text{w}^*\text{-lim}_{t\to\infty} t^{-1}S^*(t)x^*=Qx^*$, showing that $D(Q)$ is strongly closed, and so is $N(Q)$.

Since $\boldsymbol{F}^*$ is fixed by every $T^*(t)$, $t>0$, it is also fixed by $S^*(t)$ and so by Q. Hence $\boldsymbol{F}^*\subset R(Q)$. If $x^* \in D(Q)$, then for all $t>0$

$$\begin{aligned}T^*(t)Qx^*&=T^*(t)\text{w}^*\text{-}\lim_{u\to\infty} u^{-1}S^*(u)x^*=\text{w}^*\text{-}\lim_{u\to\infty} u^{-1}T^*(t)S^*(u)x^*\\&=\text{w}^*\text{-}\lim_{u\to\infty} u^{-1}(S^*(t+u)-S^*(t))x^*=Qx^*,\end{aligned}$$

the above lemma being used. This shows that $Q^2=Q$ and $R(Q)\subset \boldsymbol{F}^*$. Therefore Q is a projection onto $R(Q)=\boldsymbol{F}^*$. Next, Lemma 1 and condition (b) imply that for all $x^* \in X^*$ and $u>0$

$$\text{w}^*\text{-}\lim_{t\to\infty} t^{-1}S^*(t)(T^*(u)-I^*)x^*=\text{w}^*\text{-}\lim_{t\to\infty} t^{-1}(T^*(t)-I^*)S^*(u)x^*=0.$$

That is, $\boldsymbol{R}^*$ is contained in $N(Q)$, and so is s-closure $(\boldsymbol{R}^*)$.

So far we have proven the relation: $\boldsymbol{F}^*\oplus$s-closure $(\boldsymbol{R}^*)\subset D(Q)$. To complete the proof of (ii), we need to show that if for some sequence $t_n\to\infty$ the limit $x_1^*:=\text{w}^*\text{-lim}_{n\to\infty} t_n^{-1}\, S^*(t_n)x^*$ exists, then $x_1^* \in \boldsymbol{F}^*$ and $x^*-x_1^* \in$ s-closure $(\boldsymbol{R}^*)$.

In fact, by (b) and the lemma, we have

$$\begin{aligned}(T^*(t)-I^*)x_1^*&=\text{w}^*\text{-}\lim_{n\to\infty} t_n^{-1}(T^*(t)-I^*)S^*(t_n)x^*\\&=\text{w}^*\text{-}\lim_{n\to\infty} t_n^{-1}(T^*(t_n)-I^*)S^*(t)x^*=0\end{aligned}$$

for all $t>0$, i.e., $x_1^* \in \boldsymbol{F}^*$. On the other hand, the vector $S(t_n)x$, as an improper integral, is the strong limit of $(t_n/m)\sum_{k=1}^m T(kt_n/m)x$ as $m\to\infty$. This and the equivalence of w-lim and w*-lim in X^* yields that

$$\begin{aligned}x^*-x_1^*&=-\text{w}^*\text{-}\lim_{n\to\infty}(t_n^{-1}S^*(t_n)-I^*)x^*\\&=-\text{w}^*\text{-}\lim_{n\to\infty}\text{w}^*\text{-}\lim_{m\to\infty} m^{-1}\sum_{k=1}^m \{T^*(kt_n/m)-I^*\}x^*\\&=-\text{w-}\lim_{n\to\infty}\text{w-}\lim_{m\to\infty} m^{-1}\sum_{k=1}^m \{T^*(kt_n/m)-I^*\}x^*\\&\in \text{w-closure }(\boldsymbol{R}^*)=\text{s-closure }(\boldsymbol{R}^*).\end{aligned}$$

Lemma 2. *Let $T(\cdot)$ be a (C_0)-semigroup of operators on a Banach space X, and let A be its infinitesimal generator. Then*

(i) *$N(A)=\boldsymbol{F}$, $N(A^*)=\boldsymbol{F}^*$, s-closure $(R(A))=$s-closure $(\boldsymbol{R})$ and w*-closure $(R(A^*))=$w*-closure $(\boldsymbol{R}^*)$;*

(ii) *s-closure $(R(A^*))=$s-closure $(\boldsymbol{R}^*)$ provided that X is a Grothendieck space.*

Proof. The first and the third identities in (i) were proved in Lemma 5.2 of [5]. They result from the definition of A and the two equalities: $AS(t)x=T(t)x-x$ $(x \in X)$, $S(t)Ax=T(t)x-x$ $(x \in D(A))$. Using these two equalities and the fact that A is closed and densely defined, we can easily deduce that $A^*S^*(t)x^*=T^*(t)x^*-x^*$ for all

$x^* \in X^*$ and $S^*(t)A^*x^* = T^*(t)x^* - x^*$ for $x^* \in D(A^*)$, from which follow the inclusions: $N(A^*) \subset \boldsymbol{F}^*$ and $\boldsymbol{R}^* \subset R(A^*)$. On the other hand, the inclusions: $\boldsymbol{F}^* \subset N(A^*)$, $R(A^*) \subset$ w*-closure $(\boldsymbol{R}^*)$ are seen from the fact that $A^*x^* = \text{w*-lim}_{t\to 0} t^{-1}(T^*(t) - I^*)x^*$ for all x^* in $D(A^*)$ (see [2, p. 48]). Hence the second and the fourth identities in (i) hold.

If X is a Grothendieck space, then for each $x^* \in D(A^*)$

$$A^*x^* = \text{w*-}\lim_{n\to\infty} n(T^*(1/n) - I^*)x^* = \text{w-}\lim_{n\to\infty} n(T^*(1/n) - I^*)x^*$$

$$\in \text{w-closure } (\boldsymbol{R}^*) = \text{s-closure } (\boldsymbol{R}^*),$$

proving that $R(A^*) \subset$ s-closure $(\boldsymbol{R}^*)$. This with $R^* \subset R(A^*)$ leads to (ii).

Proof of Theorem 2. Suppose $T(\cdot)$ is strongly ergodic, i.e., P is a bounded operator on X. Then (a) and (b) have been proved in Theorem I of [5]. (c) is proved as follows.

It is clear that Q is now equal to P^*. Hence, by the identity $R(P) = \boldsymbol{F}$, we have

$$N(Q) = N(P^*) = R(P)^\perp = \boldsymbol{F}^\perp = \Big\{\bigcap_{t>0} N(T(t) - I)\Big\}^\perp$$

$$= \Big\{\bigcap_{t>0} {}^\perp(R(T^*(t) - I^*))\Big\}^\perp = ({}^\perp \boldsymbol{R}^*)^\perp = \text{w*-closure } (\boldsymbol{R}^*).$$

But we already have $N(Q) =$ s-closure $(\boldsymbol{R}^*)$. Therefore (c) holds.

Conversely, suppose (a)–(c) hold. From the above we have $\boldsymbol{F}^\perp =$ w*-closure $(\boldsymbol{R}^*) =$ s-closure $(\boldsymbol{R}^*)$, which together with (i) of Theorem 1 gives

$$D(P)^\perp = (\boldsymbol{F} \oplus \text{s-closure } (\boldsymbol{R}))^\perp = \boldsymbol{F}^\perp \cap \Big\{\bigcap_{t>0} (R(T(t) - I))^\perp\Big\}$$

$$= \boldsymbol{F}^\perp \cap \Big\{\bigcap_{t>0} N(T^*(t) - I^*)\Big\} = \text{s-closure } (\boldsymbol{R}^*) \cap \boldsymbol{F}^*,$$

and this set is $\{0\}$ by (ii) of Theorem 1. Since $D(P)$ is closed, it must be the whole space X and $T(\cdot)$ is strongly ergodic.

The equivalence of (c) and (c′) follows from Lemma 2.

References

[1] R. E. Atalla: On the ergodic theory of contractions. Revista Colombiana de Matemáticas, **10**, 75–81 (1976).

[2] P. L. Butzer and H. Berens: Semi-groups of Operators and Approximation. Springer-Verlag, Berlin (1967).

[3] J. Diestel: Grothendieck spaces and vector measures. Vector and Operator Valued Measures and Applications. Academic Press, New York, pp. 97–108 (1973).

[4] N. Dunford and J. T. Schwartz: Linear Operators. I: General Theory. Interscience (1958).

[5] P. Masani: Ergodic theorems for locally integrable semigroups of continuous linear operators on a Banach space. Advances in Math., **21**, 202–228 (1976).

[6] S.-Y. Shaw: Ergodic properties of operator semigroups in general weak topologies. J. Funct. Anal., **49**, 152–169 (1982).