36. *Asymptotic Expansions of Solutions of Fuchsian Hyperbolic Partial Differential Equations*

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(Communicated by Kôsaku YOSIDA, M. J. A., April 12, 1983)

In this paper, we deal with a Fuchsian hyperbolic partial differential equation (in Tahara [2], [4]) and determine concrete asymptotic expansions (as $t\to+0$) of solutions in $C^\infty((0,T)\times\boldsymbol{R}^n)$.

1. **Equation.** Let us consider a linear partial differential equation of the form

(E) $$(t\partial_t)^m u+\sum_{\substack{j+|\alpha|\leqq m\\ j<m}} a_{j,\alpha}(t,x)(t^\kappa\partial_x)^\alpha(t\partial_t)^j u=0,$$

where $(t,x)=(t,x_1,\cdots,x_n)\in[0,T)\times\boldsymbol{R}^n$, $\alpha=(\alpha_1,\cdots,\alpha_n)$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $a_{j,\alpha}(t,x)\in C^\infty([0,T)\times\boldsymbol{R}^n)$, $\kappa=(\kappa_1,\cdots,\kappa_n)$, $\kappa_i\in\boldsymbol{N}=\{1,2,\cdots\}$ and

$$(t^\kappa\partial_x)^\alpha=(t^{\kappa_1}\partial_{x_1})^{\alpha_1}\cdots(t^{\kappa_n}\partial_{x_n})^{\alpha_n}.$$

For hyperbolicity, we assume the following condition; all the roots $\lambda_i(t,x,\xi)$ $(1\leqq i\leqq m)$ of the equation (in λ)

$$\lambda^m+\sum_{\substack{j+|\alpha|=m\\ j<m}} a_{j,\alpha}(t,x)\xi^\alpha\lambda^j=0$$

are *real valued, simple* and *bounded* on $\{(t,x,\xi)\in[0,T)\times\boldsymbol{R}^n\times\boldsymbol{R}^n;\ |\xi|=1\}$. Then, (E) is one of the most fundamental models of Fuchsian hyperbolic equations discussed in Tahara [2], [4]. In [4], we have solved (E) in $C^\infty([0,T)\times\boldsymbol{R}^n)$ as characteristic Cauchy problems. But, here, we want to discuss (E) in $C^\infty((0,T)\times\boldsymbol{R}^n)$ from the view point of asymptotic analysis (as $t\to+0$).

2. **Result.** Let $\rho_1(x),\cdots,\rho_m(x)$ be the roots of the equation (in ρ)

$$\rho^m+\sum_{j<m} a_{j,(0,\dots,0)}(0,x)\rho^j=0.$$

Then, we can obtain the following result for (E) in $C^\infty((0,T)\times\boldsymbol{R}^n)$.

Theorem. *Assume that $\rho_i(x)-\rho_j(x)\notin\boldsymbol{Z}$ holds for any $x\in\boldsymbol{R}^n$ and $1\leqq i\neq j\leqq m$. Then, we have the following results.*

(1) *Any solution $u(=u(t,x))\in C^\infty((0,T)\times\boldsymbol{R}^n)$ of* (E) *can be expanded asymptotically into the form*

(*) $$u(t,x)\sim\sum_{i=1}^m\left\{\varphi_i(x)t^{\rho_i(x)}+\sum_{k=1}^\infty\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}\right\}$$

(as $t\to+0$) *for some $\varphi_i(x)$, $\varphi_{k,h}^{(i)}(x)\in C^\infty(\boldsymbol{R}^n)$. Further, such coefficients $\varphi_i(x)$, $\varphi_{k,h}^{(i)}(x)$ are uniquely determined by $u(t,x)$.*

(2) *Conversely, for any $\varphi_1(x),\cdots,\varphi_m(x)\in C^\infty(\boldsymbol{R}^n)$ we can find a solution $u(=u(t,x))\in C^\infty((0,T)\times\boldsymbol{R}^n)$ of* (E) *and coefficients $\varphi_{k,h}^{(i)}(x)\in C^\infty(\boldsymbol{R}^n)$ so that the asymptotic relation in* (1) *holds. Further, such*

a solution $u(t, x)$ *and coefficients* $\varphi_{k,h}^{(i)}(x)$ *are uniquely determined by* $\varphi_1(x), \cdots, \varphi_m(x)$.

Here, the meaning of the asymptotic relation (*) in (1) is as follows. Denote by $R_N(t, x)$ the N-th remainder term, that is,

$$R_N(t, x)=u(t, x)-\sum_{i=1}^{m}\left\{\varphi_i(x)t^{\rho_i(x)}+\sum_{k=1}^{N}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}\right\}.$$

Then, the asymptotic relation (*) above is defined by the following; for any $s>0$ and any compact subset K of $\boldsymbol{R}^n$, there is an $N_0\in \boldsymbol{N}$ such that for any $N\geqq N_0$

$$\sup_{x\in K}|\partial_t^l\partial_x^\alpha R_N(t, x)|=o(t^{s-l})$$

(as $t\to+0$) holds for any l and α.

Remark. In the case of analytic category, analogous results are already obtained in Tahara [3] for general Fuchsian type partial differential equations. Note that we can easily obtain the asymptotic expansion of the above form by developing the fundamental solutions (constructed in [3]) into formal series. See also Chi Min-You [1].

3. Example. Let us consider the Euler-Poisson-Darboux equation (see Weinstein [5]) of the form

$$\partial_t^2u-\Delta u+\frac{\alpha}{t}\partial_t u=0,$$

where $(t, x)\in[0, T)\times\boldsymbol{R}^n$, $\Delta=\partial_{x_1}^2+\cdots+\partial_{x_n}^2$ and $\alpha\in\boldsymbol{C}$. Assume that $\alpha\neq\pm1, \pm3, \pm5, \cdots$. Then, any solution $u\in C^\infty((0, T)\times\boldsymbol{R}^n)$ is characterized by the following asymptotic expansion

$$u(t, x)\sim\sum_{k=0}^{\infty}\frac{\Gamma((1+\alpha)/2)\Delta^k\varphi_1(x)}{2^{2k}\Gamma(k+1)\Gamma(k+(1+\alpha)/2)}t^{2k}$$
$$+\sum_{k=0}^{\infty}\frac{\Gamma((3-\alpha)/2)\Delta^k\varphi_2(x)}{2^{2k}\Gamma(k+1)\Gamma(k+(3-\alpha)/2)}t^{2k+1-\alpha}$$

(as $t\to+0$), where $\varphi_1(x), \varphi_2(x)\in C^\infty(\boldsymbol{R}^n)$.

Details and proofs will be published elsewhere.

References

[1] Chi, Min-You: On Fuchsian type partial differential equations. Sci. Rep. Wuhan Univ., 4, 1–6 (1977) (in Chinese).

[2] Tahara, H.: Cauchy problems for Fuchsian hyperbolic partial differential equations. Proc. Japan Acad., 54A, 92–96 (1978).

[3] ——: Fuchsian type equations and Fuchsian hyperbolic equations. Japan. J. Math., New Ser. 5, 245–347 (1979).

[4] ——: Singular hyperbolic systems, III. On the Cauchy problem for Fuchsian hyperbolic partial differential equations. J. Fac. Sci. Univ. Tokyo, Sect. IA Math., 27, 465–507 (1980).

[5] Weinstein, A.: The singular solutions and the Cauchy problem for generalized Tricomi equations. Comm. Pure Appl. Math., 7, 105–116 (1954).