34. An Example of a Complex of Linear Differential Operators of Infinite Order

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The purpose of this note is to show a finiteness theorem for a complex of linear differential operators of infinite order acting on the sheaf of holomorphic functions. The complex to be studied arises in the study of ϑ-zerovalue ([6]), and a detailed study of it is an important subject for the further development of [6]. A general result for microfunction solutions will be given in [7].

Let t denote a coordinate system on $\boldsymbol{C}$ and let P and Q respectively denote the matrix of linear differential operators given below:

$$P=\begin{bmatrix} & t \\ 4\pi\sqrt{-1}\left(t\frac{d}{dt}+\frac{1}{2}\right) & \end{bmatrix}$$

$$Q=\begin{bmatrix} & 1 \\ 4\pi\sqrt{-1}\frac{d}{dt} & \end{bmatrix}.$$

If we define Φ and Ψ by $\exp P-1$ $(=\sum_{n=1}^{\infty} P^n/n!)$ and $\exp Q-1$ $(\sum_{n=1}^{\infty} Q^n/n!)$ respectively, then we find ([6])

(1) Φ and Ψ are linear differential operators of infinite order and

(2) $$\Phi\Psi=\Psi\Phi.$$

For an open subset Ω of $\boldsymbol{C}$, we denote by $K(\Omega)^{\cdot}$ the complex

(3) $$0\longrightarrow \mathcal{O}(\Omega)^2 \xrightarrow{(\Phi,\Psi)} \mathcal{O}(\Omega)^4 \xrightarrow{\binom{-\Psi}{\Phi}} \mathcal{O}(\Omega)^2\longrightarrow 0$$

determined by Φ and Ψ, where $\mathcal{O}(\Omega)$ denotes the space of holomorphic functions defined on Ω. Let $H^j(K(\Omega)^{\cdot})$ denote its j-th cohomology group. Then we have the following

Theorem. *Let $\Omega(c)$ denote $\{t\in \boldsymbol{C}\,;\, \operatorname{Im} t>c\}$. Then*

(i) $H^0(K(\Omega(c))^{\cdot})\cong\begin{cases}\boldsymbol{C} & \text{for } c\geqq 0\\ 0 & \text{for } c<0\end{cases}$

(ii) $H^1(K(\Omega(c))^{\cdot})\cong\begin{cases}0 & \text{for } c\geqq 0\\ \boldsymbol{C} & \text{for } c<0\end{cases}$

(iii) $H^2(K(\Omega(c))^{\cdot})=0$ *for any c.*

Proof. Since Ψ is with constant coefficients, $\Psi: \mathcal{O}(\Omega)^2\to\mathcal{O}(\Omega)^2$ is surjective for any convex open subset Ω of $\boldsymbol{C}$ ([4]). Hence (iii) is obvious. In order to prove (ii), let us seek for a solution u in $\mathcal{O}(\Omega(c))^2$ of the following equations:

$$\begin{cases} \Phi u = f_1 \\ \Psi u = f_2 \end{cases} \tag{4}$$

with

$$\Psi f_1 = \Phi f_2, \tag{5}$$

where f_1 and f_2 belong to $\mathcal{O}(\Omega(c))^2$. Since $\Psi: \mathcal{O}(\Omega(c))^2 \to \mathcal{O}(\Omega(c))^2$ is surjective, we may suppose from the first that $f_2=0$. Then (5) reduces to

$$\Psi f_1 = 0. \tag{6}$$

By the "Fundamental Principle" type reasoning ([1], [2], [3], [4], [5], $\cdots$) we can verify that f_1 has the form

$$\sum_{\nu \in \boldsymbol{Z}} c_\nu e_\nu, \tag{7}$$

where

$$e_\nu = \begin{bmatrix} \exp(\pi\sqrt{-1}\,\nu^2 t) \\ 2\pi\sqrt{-1}\,\nu \exp(\pi\sqrt{-1}\,\nu^2 t) \end{bmatrix} \tag{8}$$

and c_ν is a complex number satisfying

$$|c_\nu| \leqq C_\varepsilon \exp(c+\varepsilon)\pi\nu^2 \qquad (\nu \in \boldsymbol{Z}) \tag{9}$$

with some constant C_ε for every $\varepsilon > 0$. (In what follows, we call this result "the Fundamental Principle for Ψ" for short.) Furthermore a simple calculation shows

$$\exp(nP) e_\nu = e_{n+\nu} \tag{10}$$

holds for every integer n and ν. Therefore, for $c \geqq 0$, $u = \sum_{\nu \in \boldsymbol{Z}} u_\nu e_\nu$ is a well-defined holomorphic solution of (4) on $\Omega(c)$, if we choose

$$u_\nu = \begin{cases} -\sum_{\mu=1}^{\nu} c_\mu & (\nu > 0) \\ 0 & (\nu = 0). \\ \sum_{\mu=\nu+1}^{0} c_\mu & (\nu < 0) \end{cases} \tag{11}$$

Note that u_ν given above satisfies the estimate (9) if $c \geqq 0$. This proves $H^1(K(\Omega(c))^\cdot) = 0$ for $c \geqq 0$. In case $c<0$, however, u_ν given above cannot satisfy (9). Actually we can prove that (4) cannot be solved if $f_1 = e_0$. In fact, if (4) were solvable with $f_2=0$, then the Fundamental Principle for Ψ entails that u should have the form

$$\sum_{\nu \in \boldsymbol{Z}} u_\nu e_\nu \tag{12}$$

with u_ν satisfying (9). But, then we should have, again by (10),

$$\begin{cases} u_{-1} - u_0 = 1 \\ u_{\nu-1} = u_\nu \qquad (|\nu| \geqq 1). \end{cases} \tag{13}$$

However, (9) and (13) cannot be consistent for $c<0$. Therefore (4) cannot be solved if $f_1 = e_0$. For general $f_1 = \sum_{\nu \in \boldsymbol{Z}} c_\nu e_\nu$ with c_ν satisfying (9), let us define φ by $\sum_{\nu \in \boldsymbol{Z}} c_\nu e_\nu - (\sum_{\nu \in \boldsymbol{Z}} c_\nu) e_0$ and consider the solvability of (4) for φ (with $f_2=0$). Note that $\sum_{\nu \in \boldsymbol{Z}} c_\nu$ is convergent by (9) if $c<0$. Then, by choosing u_ν so that

$$\begin{cases} u_{-1}-u_0=-\sum_{\mu\neq 0} c_\mu \\ u_{\nu-1}-u_\nu=c_\nu \qquad (|\nu|\geqq 1) \end{cases} \tag{14}$$

may hold, we find that u_ν satisfies (9) for $c<0$. Hence $u=\sum_{\nu\in Z} u_\nu e_\nu$ thus defined is a required solution of (4). This means that $\begin{bmatrix} f_1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} f_1-\varphi \\ 0 \end{bmatrix}$ belong to the same cohomology class in $H^1(K(\Omega(c))^{\cdot})$. On the other hand, $f_1-\varphi=(\sum_{\nu\in Z} c_\nu)e_0$ holds by the definition of φ. Therefore $H^1(K(\Omega(c))^{\cdot})\cong \boldsymbol{C}$ holds for $c<0$.

By combining (10) and the Fundamental Principle for Ψ, we can prove (i) in the same way. Q.E.D.

Remark. Although we have presented the result for global solutions, we can also prove the following local statement:

Let $K_t^{\cdot}$ denote the complex

$$0\longrightarrow \mathcal{O}^2_{C,t}\xrightarrow{(\Phi,\Psi)}\mathcal{O}^4_{C,t}\xrightarrow{\begin{pmatrix}-\Psi\\ \Phi\end{pmatrix}}\mathcal{O}^2_{C,t}\longrightarrow 0, \tag{15}$$

where $\mathcal{O}_{C,t}$ denotes the germ of the sheaf $\mathcal{O}_C$ at t. Then we have

(i) $H^0(K_t^{\cdot})\cong\begin{cases}\boldsymbol{C} & \text{if } \operatorname{Im} t>0\\ 0 & \text{if } \operatorname{Im} t\leqq 0\end{cases}$

(ii) $H^1(K_t^{\cdot})\cong\begin{cases}0 & \text{if } \operatorname{Im} t>0\\ \boldsymbol{C} & \text{if } \operatorname{Im} t\leqq 0\end{cases}$

and

(iii) $H^2(K_t^{\cdot})=0$ for every t.

References

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