

3. On Hirota's Difference Equations

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§ 1. The aim of this note is to exploit the operator approach [1] to soliton equations in studying the following non linear difference equation proposed by Hirota [2]:

$$(1.1) \quad \alpha f(\lambda+1, \mu, \nu) f(\lambda-1, \mu, \nu) + \beta f(\lambda, \mu+1, \nu) f(\lambda, \mu-1, \nu) \\ + \gamma f(\lambda, \mu, \nu+1) f(\lambda, \mu, \nu-1) = 0,$$

where α, β and γ are constants satisfying $\alpha + \beta + \gamma = 0$.

Hirota [2] found the difference Lax pair for (1.1), proved the existence of three soliton solutions and gave an ample list of non linear differential and/or difference equations obtained by taking suitable limits of (1.1). Among them is the KP (Kadomtsev-Petviashvili) equation which is written in Hirota's form as follows:

$$(1.2) \quad (D_1^4 + 3D_2^2 - 4D_1 D_3) \tau \cdot \tau = 0.$$

He also remarked a significant coincidence of the phase shift term in soliton solutions of the equations (1.1) and (1.2).

Here I shall give an explicit transformation which connects the hierarchy of the KP equation and that of Hirota's difference equation.

One of the striking discoveries of Mikio and Yasuko Sato [3] on the former was that it admits the characters of the general linear group as its solutions with

$$(1.3) \quad x_j = \text{trace} \frac{X^j}{j}, \quad X \in GL(N) \quad (j=1, 2, 3, \dots)$$

as the continuum variables. The transformation tells us that the latter, in a slightly modified form, admits them also as its solutions with the multiplicities of the eigenvalues of X as the discrete variables.

The transformation gives us also an operator solution to Hirota's difference equation in the sense of [1]. It reduces to operator solutions to equations in Hirota's list in the limit. Here I discuss briefly those for the two dimensional Toda lattice.

Finally I show that a similar consideration for the BKP hierarchy [1] leads us to the following discrete version:

$$(1.4) \quad \alpha f(\lambda+1, \mu, \nu) f(\lambda-1, \mu, \nu) + \beta f(\lambda, \mu+1, \nu) f(\lambda, \mu-1, \nu) \\ + \gamma f(\lambda, \mu, \nu+1) f(\lambda, \mu, \nu-1) + \delta f(\lambda+1, \mu+1, \nu+1) \\ \times f(\lambda-1, \mu-1, \nu-1) = 0,$$

where α, β, γ and δ are constants satisfying $\alpha + \beta + \gamma + \delta = 0$.

§ 2. Let us recall the operator solution to the KP hierarchy.

Let ψ_m and ψ_m^* ($m \in \mathbb{Z}$) be free fermions satisfying $[\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0$ and $[\psi_m, \psi_n^*]_+ = \delta_{mn}$. We use their Fourier transforms $\psi(k) = \sum_{n \in \mathbb{Z}} \psi_n k^n$ and $\psi^*(k) = \sum_{n \in \mathbb{Z}} \psi_n^* k^{-n}$. Introducing an infinite set of variables $x = (x_1, x_2, x_3, \dots)$ we set

$$(2.1) \quad \rho(x) = \iint \frac{dp}{2\pi ip} \frac{dq}{2\pi iq} \frac{p/q}{1-p/q} e^{-\xi(x,p) + \xi(x,q)} \psi(p) \psi^*(q)$$

where $\xi(x, k) = \sum_{j=1}^{\infty} x_j k^j$. It was shown in [1] that if g is an element of the Clifford group

$$(2.2) \quad \tau(x) = \langle : e^{\rho(x)} : g \rangle$$

satisfies the bilinear equation of the KP hierarchy.

Now we introduce two other infinite sets of variables $l = (l_1, l_2, l_3, \dots)$ and $a = (a_1, a_2, a_3, \dots)$. We relate them to x by

$$(2.3) \quad x_j = \sum_{i=1}^{\infty} l_i \frac{a_i^j}{j}.$$

Then the quadratic expression (2.1) is transformed into the following.

$$(2.4) \quad \rho(l; a) = \iint \frac{dp}{2\pi ip} \frac{dq}{2\pi iq} \frac{p/q}{1-p/q} \prod_{j=1}^{\infty} \left(\frac{1-a_j p}{1-a_j q} \right)^{l_j} \psi(p) \psi^*(q).$$

By using Wick's theorem we can show

Theorem 1. *Let g be an element of the Clifford group and set*

$$(2.5) \quad \tau(l; a) = \langle : e^{\rho(l; a)} : g \rangle.$$

We pick up a triple (l, m, n) among l_i 's and the corresponding one (a, b, c) among a_i 's. Then $\tau(l, m, n)$ ($= \tau(l; a)$) satisfies the following bilinear difference equation.

$$(2.6) \quad \begin{aligned} & a(b-c)\tau(l+1, m, n)\tau(l, m+1, n+1) \\ & + b(c-a)\tau(l, m+1, n)\tau(l+1, m, n+1) \\ & + c(a-b)\tau(l, m, n+1)\tau(l+1, m+1, n) = 0. \end{aligned}$$

In other words, by (2.3) we can transform solutions to the KP hierarchy into those to (2.6). Firstly, we have the following N soliton solution to (2.7) as was expected by Hirota [2].

$$(2.7) \quad \begin{aligned} \tau(x) = & 1 + \sum_{1 \leq i \leq N} c_i \sum_{j=1}^{\infty} \left(\frac{1-a_j q_i}{1-a_j p_i} \right)^{l_j} \\ & + \sum_{1 \leq i \leq i' \leq N} c_i c_{i'} c_{i''} \prod_{j=1}^{\infty} \left(\frac{1-a_j q_i}{1-a_j p_i} \right)^{l_j} \left(\frac{1-a_j q_{i'}}{1-a_j p_{i'}} \right)^{l_j} + \dots, \end{aligned}$$

where $c_{i''} = (p_i - p_{i'}) (q_i - q_{i'}) / (p_i - q_{i'}) (q_i - p_{i'})$. Secondly, taking l to be integers, we have

Theorem 2. *Let Y be a Young diagram. Let (a, b, c) be three distinct eigenvalues of an element X in $GL(N)$ with a sufficiently large integer N , and let (l, m, n) be their multiplicities. The character corresponding to Y solves the equation (2.6) with respect to the variation of (l, m, n) .*

The correspondence between (1.1) and (2.7) is given by

$$(2.8) \quad \lambda = m + n + 1, \quad \mu = n + l + 1, \quad \nu = l + m + 1,$$

$$(2.9) \quad \alpha = a(b - c), \quad \beta = b(c - a), \quad \gamma = c(a - b)$$

and

$$(2.10) \quad \tau(l, m, n) = f(m + n, n + l, l + m).$$

§ 3. Let us consider the following limit $\varepsilon \rightarrow 0$ in (2.7).

$$(3.1) \quad \begin{aligned} a &= \infty, & b &= \varepsilon, & c &= \varepsilon^{-1}, \\ l + n - 1 &= s, & m\varepsilon &= x, & n\varepsilon &= y. \end{aligned}$$

The resulting equation is called the two dimensional Toda equation [2]:

$$(3.2) \quad \begin{aligned} D_x D_y f(s, x, y) \cdot f(s, x, y) \\ = f(s, x, y)^2 - f(s - 1, x, y)f(s + 1, x, y). \end{aligned}$$

By taking $a_j = 0$ ($a_j \neq a, b, c$) the quadratic expression (2.5) is scaled to

$$(3.3) \quad \rho(s, x, y) = \iint \frac{dp}{2\pi ip} \frac{dq}{2\pi iq} \frac{(p/q)^{s+1}}{1-p/q} \frac{e^{qx+q^{-1}y}}{e^{px+p^{-1}y}} \psi(p) \psi^*(q).$$

Thus we have

Theorem 3. *If g is an element of the Clifford group, then*

$$(3.4) \quad \tau(s, x, y) = \langle : e^{\rho(s, x, y)} : g \rangle$$

solves the equation (3.2).

§ 4. Now we consider the BKP hierarchy [1]. Let ϕ_n ($n \in \mathbb{Z}$) be neutral free fermions satisfying $[\phi_m, \phi_n]_+ = (-)^m \delta_{m, -n}$. We use the Fourier transform $\phi(k) = \sum_{n \in \mathbb{Z}} \phi_n k^n$. In this section we set $x = (x_1, x_3, x_5, \dots)$,

$$(4.1) \quad \tilde{\xi}(x, k) = x_1 k + x_3 k^3 + x_5 k^5 + \dots$$

and

$$(4.2) \quad \rho_B(x) = \iint \frac{dp}{2\pi ip} \frac{dq}{2\pi iq} \frac{q/p}{1+q/p} e^{-\tilde{\xi}(x, p) - \tilde{\xi}(x, q)} \phi(p) \phi(q).$$

Then, if g is an element of the Clifford group

$$(4.3) \quad \tau_B(x) = \langle : e^{\rho_B(x)} : g \rangle$$

solves the BKP hierarchy.

We set

$$(4.4) \quad x_j = 2 \sum_{i=1}^{\infty} l_i \frac{a_i^j}{j}.$$

Then $\rho_B(x)$ reduces to

$$(4.5) \quad \begin{aligned} \rho_B(l, a) \\ = \iint \frac{dp}{2\pi ip} \frac{dq}{2\pi iq} \frac{q/p}{1+q/p} \prod_{j=1}^{\infty} \left(\frac{1-a_j p}{1+a_j p} \right)^{l_j} \left(\frac{1-a_j q}{1+a_j q} \right)^{l_j} \phi(p) \phi(q). \end{aligned}$$

Then we have

Theorem 4. *Let g be an element of the Clifford group. We set*

$$(4.6) \quad \tau_B(l, a) = \langle : e^{\rho_B(l, a)} : g \rangle.$$

We pick up a triple (l, m, n) among l_j 's and the corresponding one (a, b, c) among a_j 's. Then $\tau(l, m, n)$ ($\stackrel{\text{def}}{=} \tau_B(l, a)$) solves the following bilinear difference equation.

$$(4.7) \quad \begin{aligned} (a+b)(a+c)(b-c)\tau(l+1, m, n)\tau(l, m+1, n+1) \\ + (b+c)(b+a)(c-a)\tau(l, m+1, n)\tau(l+1, m, n+1) \end{aligned}$$

$$\begin{aligned}
&+(c+a)(c+b)(a-b)\tau(l, m, n+1)\tau(l+1, m+1, n) \\
&+(a-b)(b-c)(c-a)\tau(l+1, m+1, n+1)\tau(l, m, n)=0.
\end{aligned}$$

We also note that the BKP equation

$$(4.8) \quad (D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5)\tau \cdot \tau = 0$$

is recovered in a suitable limit of (4.7).

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References

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