123. *A Note on Non-Singular Morse-Smale Flows on S^3**)*

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In this note, we prove the following

Theorem 1.**) *Every non-singular Morse-Smale flow on S^3 has at least two unknotted closed orbits.*

For definitions of *non-singular Morse-Smale flows* (*NMS* for short) and *round handle decompositions* (*RHD* for short) and relations between them, see Morgan [1]. A closed orbit of an NMS of S^3 is *attracting, hyperbolic* and *repelling* if the dimension of the unstable manifold is equal to one, two and three respectively. A hyperbolic closed orbit is *untwisted* if the unstable manifold is orientable and *twisted* otherwise.

If an NMS has twisted hyperbolic closed orbits, it can be changed near them as shown in Fig. 1, so that the new flow becomes an NMS without twisted hyperbolic closed orbits and the link consisting of attracting and repelling closed orbits of the new flow is a sublink of the link of all closed orbits of the old NMS.

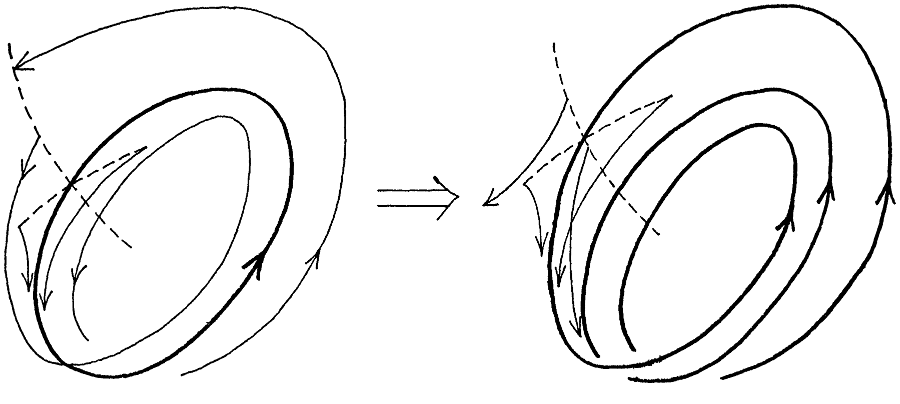

Fig. 1

This says that Theorem 1 above is a corollary to

Theorem 2. *Every non-singular Morse-Smale flow on S^3 without twisted hyperbolic closed orbits has at least two unknotted closed*

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**) M. Wada recently obtained a characterization of links which are closed orbits of NMS's on S^3, which includes this result as a corollary.

orbits, each of which is attracting or repelling.

In the sequel, hyperbolic closed orbits, hence round 1-handles are assumed to be untwisted. Then there are the following 11 types of the ways of attaching round 1-handles to the boundary consisting of tori.

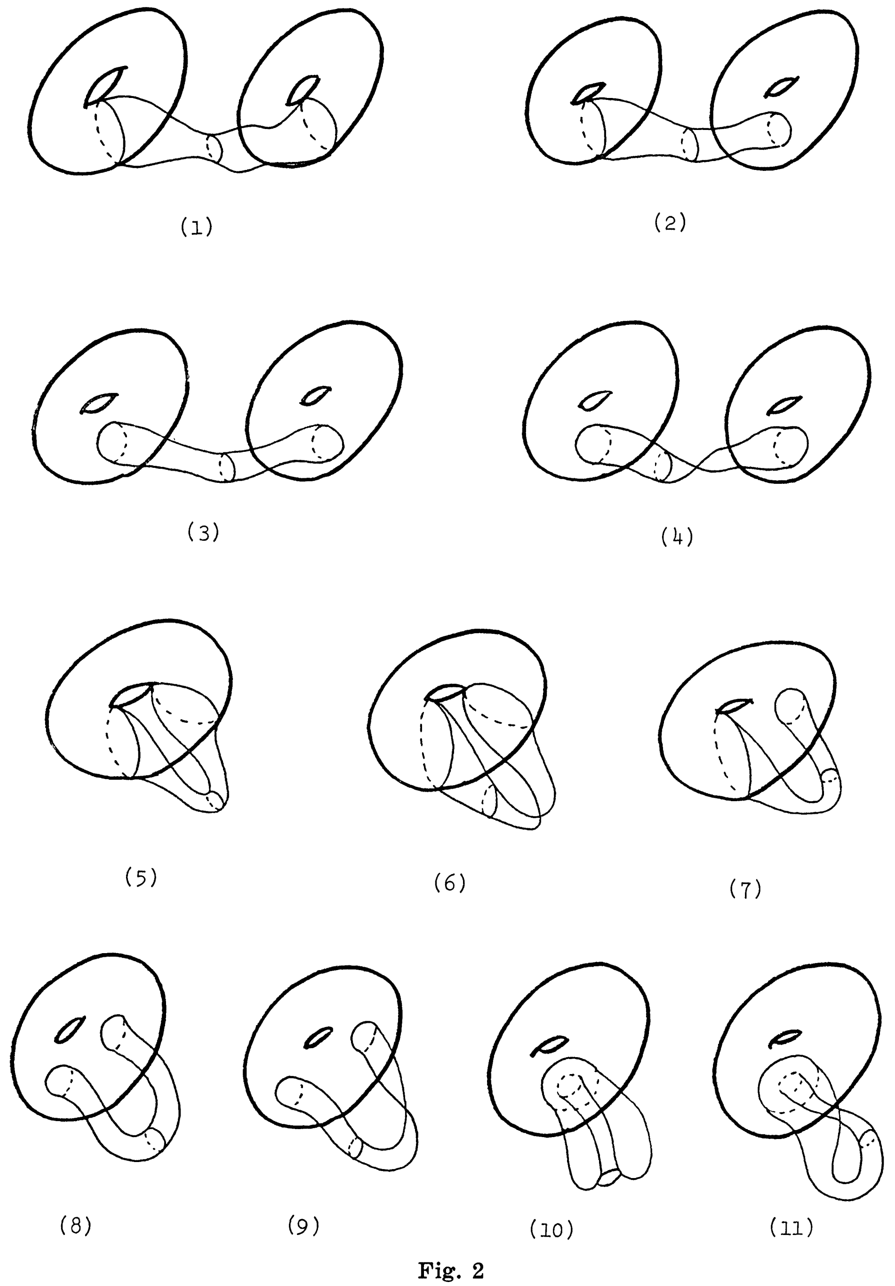

Fig. 2

Lemma 1. *Every round* 1-*handle of an RHD of* S^3 *is of types* (1), (2), (4), (5), (7) *or* (10).

Proof. Types (3) and (8) yield embedded spheres transverse to the flow and types (6), (9) and (11) yield embedded Klein bottles, and thus these types are excluded. Other ways of attaching imply that the boundary of the resulting manifold consists again of tori. Thus, by induction, we get the lemma.

For a given RHD of S^3, we can fatten up round 1-handles, i.e. affix collars of components of boundary tori which it is actually attached to, and then get a new decomposition of S^3 (Morgan [1]). Topological types of fattened round 1-handles are as follows: (two punctured disk)$\times S^1$ for a round 1-handle of type (1) or (5), $T^2\times I \sharp S^1\times D^2$ for type (2) or (10), $T^2\times I \sharp T^2\times I$ for type (4), and $S^1\times D^2 \sharp S^1\times D^2$ for type (7).

We say that a decomposition $\mathcal{D}$ of S^3 (or its 3-dimensional compact submanifold) is *RHD-like* if $\mathcal{D}$ is obtained by cutting S^3 (or submanifold) along disjointly embedded tori, and each connected component of $\mathcal{D}$ is either a solid torus or one of the manifolds listed above. Then Theorem 2 is reduced to

Proposition. *Every RHD-like decomposition* $\mathcal{D}$ *of* S^3 *contains at least two unknotted solid tori.*
We let $d(\mathcal{D})$ denote the number of components of $\mathcal{D}$ which are not solid tori, and prove this proposition by induction on $d(\mathcal{D})$.

Lemma 2. *If* $\mathcal{D}$ *consists of three solid tori and one* (*two punctured disk*)$\times S^1$, *then the proposition holds.*

Proof. The natural S^1-action on (two punctured disk)$\times S^1$ extends to whole of S^3. If this action has a fixed point, then every S^1-orbit and the fixed point set is unknotted in S^3 and thus the lemma follows. Otherwise, it gives a Seifert fibred structure of S^3. Since any singular fibre is the core of some solid torus of $\mathcal{D}$, the classification of Seifert fibred structures of S^3 implies the result.

Proof of Proposition. This is obvious if $d(\mathcal{D})=0$. Suppose now that this proposition is verified when $d(\mathcal{D})\leq k$. Then we have

Assertion. *Let* $S^1\times D^2\subset S^3$ *be an embedded solid torus and* $\mathcal{D}'$ *be an RHD-like decomposition of* S^3-int$(S^1\times D^2)$ *with* $d(\mathcal{D}')\leq k$. *Then, if* $S^1\times D^2$ *is unknotted,* $\mathcal{D}'$ *contains at least one solid torus unknotted in* S^3 *and if* $S^1\times D^2$ *is knotted,* $\mathcal{D}'$ *contains at least two unknotted solid tori.*
Let $\mathcal{D}$ be an RHD-like decomposition of S^3 with $d(\mathcal{D})=k+1$ and take a component $H\in\mathcal{D}$ which is not a solid torus.

Case 1. $H\cong$(*two punctured disk*)$\times S^1$. Let A_1, A_2 and A_3 denote the components of S^3-int H. Since ∂A_1 is a torus, either A_1 is a solid torus or $H\cup A_2\cup A_3$ is a knotted solid torus. In the latter case, Assertion implies that A_1 contains at least two unknotted solid tori. Applying this argument also to A_2 and A_3, we can assume that each A_i is a solid torus. Then Lemma 2 says that at least two of A_i's are unknotted. Again by Assertion, we know that each unknotted A_i contains at least one unknotted solid torus in S^3, and thus we are done.

Case 2. $H \cong T^2 \times I \# S^1 \times D^2$. There is a component B of S^3-int H such that $H \cup B$ is $T^2 \times I$. We can delete components in $H \cup B$ from $\mathcal{D}$ and get a new RHD-like decomposition $\mathcal{D}'$ of S^3 with $d(\mathcal{D}') \leq k$. Thus we are done.

Case 3. $H \cong T^2 \times I \# T^2 \times I$. Same as in Case 2.

Case 4. $H \cong S^1 \times D^2 \# S^1 \times D^2$. Let C_1 and C_2 denote the components of S^3-int H. Since $H \cup C_2$ is a solid torus, Assertion implies that C_1 contains at least one unknotted solid torus. So does C_2 by the same reason and thus we are done.

This completes the induction and we get the required result.

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Reference

[1] J. Morgan: Non-singular Morse-Smale flows on 3-dimensional manifolds. Topology, **18**, 41–53 (1979).