122. *The Mean Square of Dirichlet L-Functions*

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§ 1. **Statement of results.** Let q be a positive integer, χ a primitive Dirichlet character mod q, and $L(s,\chi)$ the corresponding Dirichlet L-function. The purpose of this article is to show the following asymptotic formula:

Theorem 1. *If q is odd and $T\geqq 1$, then for any $\varepsilon>0$,*

$$(1)\quad \int_1^T |L(1/2+it,\chi)|^2\,dt = q^{-1}\varphi(q)T\cdot\log T - q^{-1}\varphi(q)\Big\{1+\log 2\pi-2\gamma-\log q-2\sum_{p|q}(p-1)^{-1}\log p\Big\}T + 4(2\pi/q^3)^{1/2}\Big(\sum_{n=1}^{q-1}{}' n\Big)T^{1/2} + O((qT)^{1/3+\varepsilon}+q^2(qT)^{1/4}\log(qT)+q^{5/2}(qT)^{1/8}+q^{9/2}\log^2(qT)),$$

where $\varphi(q)$ is Euler's function, γ is Euler's constant, and, also throughout this paper, the symbol $\sum_n'$ indicates the sum in which n runs over the values with $(n,q)=1$.

The following corollary is easily deduced from Theorem 1, using Lemma 3 of Heath-Brown [4]. Although he proved this lemma only for the case of the Riemann zeta-function, we can immediately generalize the statement to L-functions if $t\gg q$.

Corollary. *If q is odd and $t\gg q^{23}$, then for any $\varepsilon>0$,*

$$(2)\quad L(1/2+it,\chi)=O((qt)^{1/6+\varepsilon}).$$

As a consequence of Kolesnik's result [5], we can show that (2) holds if $t\gg q^{72+\varepsilon}$. Our result covers the range $q^{23}\ll t\ll q^{72}$. (See also [2] and [3].)

Theorem 1 is a generalization of the following formula, which is proved by Balasubramanian [1]:

$$\int_1^T |\zeta(1/2+it)|^2\,dt = T\cdot\log T-(1+\log 2\pi-2\gamma)T+O(T^{1/3+\varepsilon}),$$

where $\zeta(s)$ is the Riemann zeta-function. Our proof of Theorem 1 is an analogue of Balasubramanian's argument. But if we modify his calculation directly, we get only an error term which is rather bad with respect to q. Therefore, in the last stage, we utilize Heath-Brown's estimate of some type of exponential sums [3], which is based on Weil's famous estimate of Kloosterman's sum [9].

For even q, we can only get the following weaker result:

Theorem 2. *If we replace the error term of the right-hand side* *of* (1) *by*

$$O(q^{7/12}T^{5/12}\log^2(qT)+q^{3/2}T^{5/12}\log(qT)+q^{19/12}T^{5/12}$$
$$+q^{3/4}T^{1/4}\log^{3/2}T+q\cdot\log^2(qT)+q^2\log(qT)),$$

then, this asymptotic formula holds for any positive integer q.

We can prove Theorem 2 by a similar generalization of Titchmarsh's argument [8], instead of [1]. We can't apply the argument of pp. 199–200 of [8] for even q, but we can go through the obstacle by using the Pólya-Vinogradov inequality.

We remark that the third term of the right-hand side of (1) is a new aspect; it vanishes in the case of the zeta-function. Theorem 2 asserts that this term appears for every $q\geqq 2$. (Here we mention that the work of Narlikar [6] also suggests the existence of this term.) The proof of Theorem 2 is in fact simpler than that of Theorem 1, so in this paper, we only show (the outline of) the proof of Theorem 1.

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§ 2. Sketch of the proof of Theorem 1. Let

$$C=C(\chi)=\sum_{n=1}^{q}\chi(n)\exp(-2\pi in/q),$$
$$a=(1-\chi(-1))/2,\quad \rho=i^{-(1/2)a}C^{-1/2}q^{1/4},\quad k=q[(t/2\pi q)^{1/2}],$$
$$\eta=(qt/2\pi)^{1/2}-k-1/2,\qquad \exp(i\delta)=\rho\exp(i\pi(2a-1)/8),$$
$$\exp(i\alpha_n)=\chi(n)\ \text{ if }\ (n,q)=1,\ \text{ and},$$
$$\exp(i\vartheta)=\rho(q/\pi)^{(1/2)s-1/4}(\Gamma((s+a)/2)/\Gamma((1-s+a)/2))^{1/2},$$

where, $\Gamma(s)$ is the Γ-function and [ ] is the usual Gauss symbol. Then, from Theorem 6 of [7], we have

$$\begin{aligned}\exp(i\vartheta)L(1/2+it,\chi)&=2\sum_{n=1}^{k}{}' n^{-1/2}\cos(\vartheta+\alpha_n-t\cdot\log n)\\&\quad+(2\pi/qt)^{1/4}(-1)^{k-1}(\sin(2\pi(\eta+(1/4)q)))^{-1}\\&\qquad\times\sum_{n=1}^{q}{}'\sin(2\pi q^{-1}(\eta+(q+1)/2-n)^2-\pi n^2q^{-1}+\alpha_n+\delta)\\&\quad+O(q^{-1/4}t^{3/4})\\&=f_1(t)+f_2(t)+f_3(t),\ \text{say.}\end{aligned}$$

So,

$$\begin{aligned}\int_1^T|L(1/2+it,\chi)|^2\,dt&=\int_1^T f_1(t)^2dt+\int_1^T f_2(t)^2dt+\int_1^T f_3(t)^2dt\\&\quad+2\int_1^T f_1(t)f_2(t)\,dt+2\int_1^T f_2(t)f_3(t)\,dt+2\int_1^T f_3(t)f_1(t)\,dt.\end{aligned}$$

We carry out the evaluation of this right-hand side in an analogous way to the method of [1], Part I. In this calculation, we encounter some different situations from the case of the zeta-function. In particular, Lemma 11 (ii) and (iii) of [1] no longer holds, so we cannot apply Balasubramanian's argument of simplifying A_{14} to general q.

(See [1] p. 561.) But, if $q\,(\geqq 3)$ is odd, we can fortunately estimate the corresponding term by the method of pp. 199–200 of Titchmarsh [8]. Consequently, corresponding to Lemma 21 of [1], we can show that

$$(3)\quad \int_1^T |L(1/2+it,\chi)|^2\,dt = q^{-1}\varphi(q)T\cdot\log T - q^{-1}\varphi(q)\Big(1+\log 2\pi - 2\gamma - \log q - 2\sum_{p|q}(p-1)^{-1}\log p\Big)T - 4(2\pi q^{-3})^{1/2}\Big(\sum_{n=1}^{q-1}{}' n\Big)T^{1/2} + 4U + 4B + O(T^{1/4}\log^{1/2}(qT) + q^2(qT)^{1/4} + q^{5/2}(qT)^{1/8} + q^{9/2}\log^2(qT)),$$

where the definition of U and B is as follows: If we put

$$\vartheta_1(t) = (1/2)t\cdot\log(qt/2\pi) - (1/2)t + (1/4)i\cdot\log(q^{-1}C^2(\chi)) - \pi/8$$

and $L = q[(T/2\pi q)^{1/2}]$, then

$$U = \sum_{1\leqq m<n\leqq L}{}'\sum{}' (mn)^{-1/2}\log^{-1}(m^{-1}n)\sin(\alpha_m - \alpha_n + T\cdot\log(m^{-1}n)),$$

$$B = \sum_{1\leqq m<n\leqq L}{}'\sum{}' (mn)^{-1/2}(2\vartheta_1'(T) - \log(mn))^{-1} \times \sin(2\vartheta_1(T) + \alpha_m + \alpha_n - T\cdot\log(mn)).$$

For the purpose of estimating U and B, we consider the following sum. For $\delta > 0$, we define

$$X = \sum_{k<(M/H)^{1+\delta}}\ \sum_{M<m\leqq 2M} (m(m-k))^{-1/2}\log^{-1}(m^{-1}(m-k)) \times \overline{\chi(m)}\chi(m-k)\exp(it\cdot\log(m^{-1}(m-k))).$$

We can reduce the estimate of X to that of the sum

$$S = S(q;\chi,-k,n) = \sum_{m=0}^{q-1}\chi(m-k)\overline{\chi(m)}\exp(2\pi imn/q).$$

Applying Heath-Brown's estimate of S (Lemmas 5–8 of [3]), we can show $X \ll H$ if $H \gg (qT)^{1/3+\varepsilon}$ for arbitrary small $\varepsilon > 0$. Then, by "the multiple integration process" of Balasubramanian [1], we can deduce the estimate $U = O((qT)^{1/3+\varepsilon})$. Finally, we can treat B in a similar way, and get the same estimate $B = O((qT)^{1/3+\varepsilon})$. These estimates and (3) complete the proof of Theorem 1.

References

[1] Balasubramanian, R.: An improvement on a theorem of Titchmarsh on the mean square of $|\zeta(1/2+it)|$. Proc. London Math. Soc., (3) **36**, 540–576 (1978).

[2] Fujii, A., Gallagher, P. X., and Montgomery, H. L.: Some hybrid bounds for character sums and Dirichlet L-series. Topics in Number Theory, Colloq. János Bolyai Math. Soc. Debrecen, pp. 41–57 (1974).

[3] Heath-Brown, D. R.: Hybrid bounds for Dirichlet L-functions. Invent. math., **47**, 149–170 (1978).

[4] ——: The twelfth power moment of the Riemann-function. Quart. J. Math. Oxford, (2) **29**, 443–462 (1978).

[5] Kolesnik, G.: On the order of Dirichlet L-functions. Pacific J. Math., **82**, 479–484 (1979).

[6] Narlikar, M. J.: On the mean square value of Hurwitz zeta function. Proc. Indian Acad. Sci., Math. Sci., **90**, 195–212 (1981).

[7] Siegel, C. L.: Contribution to the theory of the Dirichlet L-series and Epstein zeta-functions. Ann. of Math., **44**, 143–172 (1943).

[8] Titchmarsh, E. C.: On van der Corput's method and the zeta-function of Riemann (V). Quart. J. Math., Oxford, **5**, 195–210 (1934).

[9] Weil, A.: On some exponential sums. Proc. Nat. Acad. Sci. USA, **34**, 204–207 (1948).