120. On Spectral Families of Projections in Hardy Spaces

By Earl BERKSON
Department of Mathematics, University of Illinois

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1. **Introduction.** It is well-known that outside the L^2 setting many major aspects of classical analysis cannot be treated by the projection-valued measures in abstract spectral theory. However, the notions of well-bounded operator and spectral family (introduced in [1], [2]) afford an approach to abstract operator theory using Riemann-Stieltjes integrals and divorced from vector measures. Recently in [1] the scope of these notions has been considerably expanded. In particular, [1, Theorem (4.20)] (see (2.1) below) affords an abstract operator-theoretic rationale for Fourier inversion in classical reflexive L^p spaces [1, (4.47)]. In a forthcoming paper [2] (outlined in this note) we show that the foregoing circle of ideas can be applied to complex analysis. Our main result is that every strongly continuous one-parameter group of isometries on $H^p(\boldsymbol{D})$, where $\boldsymbol{D}$ is the open unit disc in $\boldsymbol{C}$ and $1<p<\infty$, has a spectral decomposition as in the conclusion of the generalized Stone's theorem (see (3.1) below). One isometric group, the parabolic group $\{V_t^{(p)}\}$ in Theorem (3.4) below, is of special interest. Its spectral family corresponds to the M. Riesz projections restricted to $H^p(\boldsymbol{R})$ ($\boldsymbol{R}$ is the real line). The spectral family of $\{V_t^{(p)}\}$ is concretely described in (3.6) below. A pleasant by-product of the parabolic case is the incorporation of a key ingredient of the Paley-Wiener theorem for $1<p\leq 2$ into the abstract framework of the generalized Stone's theorem (see (3.5) below). For a condensed account of the general theory of well-bounded operators and spectral families see [1, § 2].

2. **Abstract preliminaries. Definition.** A *spectral family* in a Banach space X is a projection-valued function $E(\cdot): \boldsymbol{R}\to\mathscr{B}(X)$ such that: (i) $E(\cdot)$ is uniformly bounded, monotone increasing, and strongly right continuous on $\boldsymbol{R}$; (ii) $E(\cdot)$ has a strong left-hand limit at each point of $\boldsymbol{R}$; and (iii) $E(\lambda)\to 0$ (resp., $E(\lambda)\to I$) strongly as $\lambda\to-\infty$ (resp., $\lambda\to+\infty$).

(2.1) **Generalized Stone's theorem** ([1, Theorem (4.20)]). *Let $\{T_t\}$ be a strongly continuous one-parameter group of operators on the Banach space X with infinitesimal generator $\mathcal{S}$. Suppose that*: (a) *for each $t\in\boldsymbol{R}$, $T_t=e^{iA_t}$, where A_t is a well-bounded operator of type* (*B*) *whose spectrum, $\sigma(A_t)$, is contained in* $[0,2\pi]$; *and* (b) $\sup\{\|E_t(\lambda)\|: t, \lambda$

$\in \boldsymbol{R}\} < \infty$, *where* $E_t(\cdot)$ *is the spectral family of* A_t. *Then*:

(i) *There is a unique spectral family* $\mathcal{E}(\cdot)$ *in* X *(called the Stone-type spectral family of* $\{T_t\}$*) such that*

$$T_t x = \lim_{a \to +\infty} \int_{-a}^{a} e^{it\lambda} d\mathcal{E}(\lambda)x \qquad \text{for } t \in \boldsymbol{R},\ x \in X.$$

(ii) *The commutants of* $\{T_t : t \in \boldsymbol{R}\}$ *and* $\{\mathcal{E}(\lambda) : \lambda \in \boldsymbol{R}\}$ *are equal.*

(iii) *The domain of* $\mathcal{S}$, $\mathcal{D}(\mathcal{S})$, *equals*

$$\left\{x \in X : \lim_{a \to +\infty} \int_{-a}^{a} \lambda d\mathcal{E}(\lambda)x \text{ exists}\right\},$$

and

$$\mathcal{S}(x) = i \lim_{a \to +\infty} \int_{-a}^{a} \lambda d\mathcal{E}(\lambda)x \qquad \text{for } x \in \mathcal{D}(\mathcal{S}).$$

Remark. It is not difficult to see that $\{T_t\}$ must be uniformly bounded, and $\sigma(\mathcal{S})$ is pure-imaginary.

(2.2) **Theorem** ([2, Theorem (4.4)]). *If, in addition to the hypotheses of Theorem* (2.1), $\sigma(\mathcal{S}) \subseteq \{i\lambda : \lambda \le 0\}$, *then* $\mathcal{E}(\lambda) = I$ *for* $\lambda \ge 0$.

The proof of Theorem (2.1) in [1] gives a representation for $\mathcal{E}(\cdot)$. This allows one to show the following in [2].

(2.3) **Theorem.** *Suppose* $\{T_t\}$ *satisfies the hypotheses of Theorem* (2.1), *and* M *is a closed subspace of* X *invariant under* $\{T_t\}$. *Then* M *is* $\mathcal{E}(\cdot)$*-invariant, and the group of restrictions* $\{T_t | M\}$ *satisfies the hypotheses of Theorem* (2.1) *with Stone-type spectral family* $\mathcal{E}(\cdot) | M$.

3. Groups of isometries on reflexive Hardy spaces. (3.1) **Main Theorem.** *If* $\{T_t\}$ *is a strongly continuous one-parameter group of isometries on* $X = H^p(\boldsymbol{D})$, $1 < p < \infty$, *then conclusions* (2.1) (i), (ii), *and* (iii) *hold.*

Sketch of Proof. We can assume $p \ne 2$. In view of [4, Theorem (2.4)] and [3, Theorem (2.1)], the proof reduces to the following situations: $\{T_t\}$ is the translation group of L^p of the circle $\boldsymbol{T}$ restricted to $H^p(\boldsymbol{T})$ (elliptic case); or $T_t f = (\phi_t')^{1/p} f(\phi_t)$, where $\{\phi_t\}$, $t \in \boldsymbol{R}$, is a certain hyperbolic group of Möbius transformations of $\boldsymbol{D}$ or the parabolic group $\{\eta_t\}$, where

(3.2) $\eta_t(z) \equiv [(1 - i2^{-1}t)z + i2^{-1}t] / (-i2^{-1}tz + 1 + i2^{-1}t)$.

All three cases can be treated by combining [5, Theorem 1] with Theorem (2.3). We consider only the parabolic case here. Let $W^{(p)}$ be the standard isometry of $H^p(\Pi^+)$ onto $H^p(\boldsymbol{D})$, where Π^+ is the right half-plane. For $f \in H^p(\Pi^+)$, we write $\Omega^{(p)} f$ for the boundary function of f. Then it is easy to see that in the parabolic case

(3.3) $\Omega^{(p)}[W^{(p)}]^{-1} T_t W^{(p)} [\Omega^{(p)}]^{-1} = \mathcal{U}_t^{(p)} | H^p(\boldsymbol{R}), \qquad \text{for } t \in \boldsymbol{R}$,

where $\{\mathcal{U}_t^{(p)}\}$, $t \in \boldsymbol{R}$, is the translation group on $L^p(\boldsymbol{R})$.

The parabolic case forms the subject matter of our remaining considerations. For $1 < p < \infty$ ($p = 2$ is no longer excluded), we let $E^{(p)}(\cdot)$ denote the Stone-type spectral family of $\{\mathcal{U}_t^{(p)}\}$. It is shown in

[1, (4.47) (ii)] that for $\lambda \in \boldsymbol{R}$, $E^{(p)}(\lambda)$ is the M. Riesz projection for λ (i.e., the multiplier operator on $L^p(\boldsymbol{R})$ corresponding to the characteristic function of $(-\infty, \lambda]$). Let $E_p(\cdot)=E^{(p)}(\cdot)|H^p(\boldsymbol{R})$. From (3.3) we have the following theorem.

(3.4) **Theorem.** *For* $1<p<\infty$, *let* $\{V_t^{(p)}\}$ *be the group given by*

$$V_t^{(p)}f=(\eta_t')^{1/p}f(\eta_t) \qquad \text{for } t \in \boldsymbol{R},\ f \in H^p(\boldsymbol{D}).$$

Then $F_p(\cdot)$, *the Stone-type spectral family of* $\{V_t^{(p)}\}$, *is given by*

$$F_p(\lambda)=W^{(p)}[\Omega^{(p)}]^{-1}E_p(\lambda)\Omega^{(p)}[W^{(p)}]^{-1} \qquad \text{for } \lambda \in \boldsymbol{R}.$$

One component of the Paley-Wiener theorem for $1<p\leq 2$ is that for $f \in H^p(\boldsymbol{R})$ $\check{f}$ vanishes for almost all $\lambda<0$. This fact follows from the next theorem.

(3.5) **Theorem.** *For* $1<p<\infty$, $E_p(\lambda)=I$ *for* $\lambda\geq 0$.

Sketch of Proof. For Re $\zeta\geq 0$, let $\mathcal{T}_\zeta^{(p)}$ be the corresponding translation operator on $H^p(\Pi^+)$. Then $\{\mathcal{T}_\zeta^{(p)}\}$ is a strongly continuous semigroup of contraction operators on Re $\zeta>0$, holomorphic on Re $\zeta>0$. Use [6, Theorem 17.9.2] and Theorem (2.2) to complete the proof.

We now give a concrete description of the family $F_p(\cdot)$ in (3.4).

(3.6) **Theorem.** *Let* μ *denote the function* $(1+z)(1-z)^{-1}$, *and for* $a>0$ *let* ξ_a *denote the singular inner function* $\exp(-a\mu)$. *Suppose* $1<p<\infty$. *Then* $F_p(a)=I$ *for* $a\geq 0$, *and for* $a>0$:

(i) $F_p(-a)H^p(\boldsymbol{D})=\xi_a H^p(\boldsymbol{D})$;

(ii) *the null space of* $F_p(-a)$ *is the closed linear manifold in* $H^p(\boldsymbol{D})$ *spanned by the following set of functions*

$$\{(1-z)^{-2/p}[\xi_a(z)-1][2\pi i n a^{-1}-\mu(z)]^{-1} : n=0, \pm 1, \pm 2, \cdots\}.$$

The proof of (3.6) (ii) is lengthy; its essence is to use a suitable regularization argument on Fourier transforms to show that $\mathcal{S}_{p,a}$ is dense in $N_{p,a}$, where $N_{p,a}=\ker E_p(-a)$, and

$$\mathcal{S}_{p,a}=N_{p,a}\cap\{f \in H^2(\boldsymbol{R}) : \check{f} \in C^\infty(\boldsymbol{R}) \text{ and support } \check{f}\subseteq[0, a]\}.$$

Remarks. Let $\mathcal{R}_p$ be the M. Riesz projection of $L^p(\boldsymbol{T})$ on $H^p(\boldsymbol{T})$ along $\overline{H_0^p(\boldsymbol{T})}$, $1<p<\infty$. For $a>0$, let $P_p(-a)f=\xi_a\mathcal{R}_p(\bar{\xi}_a f)$ for $f \in H^p(\boldsymbol{T})$, and set $P_p(a)=I$ for $a\geq 0$. It is shown in [2, § 6] that $P_p(\cdot)$ is a spectral family such that $P_p(-a)H^p(\boldsymbol{D})=\xi_a H^p(\boldsymbol{D})$ for $a>0$, but $P_p(\cdot)$ equals $F_p(\cdot)$ if and only if $p=2$.

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