115. *Eigenvalues of the Laplacian on Wildly Perturbed Domain*

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We remove m balls with centers $\{w_i^{(m)}\}_{i=1}^m$ and radius α/m from a bounded domain Ω in $\boldsymbol{R}^3$ with smooth boundary γ. If m balls are dispersed in a specific configuration as $m\to\infty$, then we can give a precise asymptotic behaviour of the k-th eigenvalue of the Laplacian in $\Omega\setminus\overline{m \text{ balls}}$ under the Dirichlet condition on its boundary. Our method is based on perturbational calculus.

By $w(m)$ we denote $\{w_i^{(m)}\}_{i=1}^m$. A sequence $\{w(m)\}_{m=1}^\infty$ satisfying the following conditions (C-1), (C-2) is said to be of class $\mathcal{O}$:

(C-1) There exists a constant $\tilde{C}>0$ independent of m such that

$$|w_i^{(m)}-w_j^{(m)}|\geq \tilde{C}m^{-1/3} \qquad (j\neq i)$$

$$\mathrm{dist}\,(w_j^{(m)}, \boldsymbol{R}^3\setminus\Omega)\geq \tilde{C}m^{-1/3} \qquad (1\leq j\leq m).$$

(C-2) Fix $0<p\leq 1$. Then, there exists a constant C_p independent of m such that

$$\left|\frac{1}{m}\sum_{j=1}^m f(w_j^{(m)})-\int_\Omega f(x)V(x)dx\right|\leq C_p m^{-p/3}\|f\|_{C^p(\Omega)}$$

holds for any $f\in C^p(\Omega)$. Here $V(x)$ is a non-negative C^1 function on $\bar{\Omega}$ satisfying

$$\int_\Omega V(x)dx=1.$$

Moreover,

$$\max_j\left|\frac{1}{m}\sum_{\substack{1\leq i\leq m\\ i\neq j}}\frac{f(w_i^{(m)})}{|w_i^{(m)}-w_j^{(m)}|}-\int_\Omega \frac{V(y)f(y)}{|y-w_j|}dy\right|\leq C_p m^{-p\sigma/3}\|f\|_{C^p(\Omega)},$$

$$(0<\sigma\leq 1).$$

We put $B(\varepsilon\,;\,w_j^{(m)})=\{x\in \boldsymbol{R}^3\,;\,|x-w_j^{(m)}|<\varepsilon\}$. Let $0<\mu_1(\varepsilon\,;\,w(m))\leq\mu_2(\varepsilon\,;\,w(m))\leq\cdots$ be the eigenvalues of $-\Delta$ $(=-\mathrm{div\ grad})$ in $\Omega_{\varepsilon,w(m)}=\Omega\setminus\overline{B(\varepsilon\,;\,w_j^{(m)})}$ under the Dirichlet condition on $\partial\Omega_{\varepsilon,w(m)}$. We arrange them repeatedly according to their multiplicities. Let μ_k^V be the k-th eigenvalue of $-\Delta+4\pi\alpha V(x)$ in Ω under the Dirichlet condition on γ. The main result of this paper is the following:

Theorem 1. *Fix $\alpha>0$. Suppose that $\{w(m)\}_{m=1}^\infty$ is of class $\mathcal{O}$. Then, $\mu_k(\alpha/m\,;\,w(m))$ tends to μ_k^V as $m\to\infty$. Moreover,*

$$|\mu_k^V-\mu_k(\alpha/m\,;\,w(m))|\leq C_{\varepsilon'}m^{\varepsilon'-\sigma/6}$$

holds, where ε' is an arbitrary small fixed positive number.

Remark. It should be remarked that the sum of the radius of

m-balls is α for any m.

We explain the main idea of our proof of Theorem 1. Let $G_m(x, y; w(m))$ be the Green function of the Laplacian in $\Omega_{\alpha/m, w(m)}$ under the Dirichlet condition on its boundary. It satisfies

$$\Delta_x G_m(x, y; w(m)) = -\delta(x-y) \qquad \text{for } x, y \in \Omega_{\alpha/m, w(m)}$$

and $G_m(x, y; w(m)) = 0$ for $x \in \partial\Omega_{\alpha/m, w(m)}$. Hereafter we abbreviate $w_i^{(m)}$ as w_i for the sake of simplicity. We put

$$h_m(x, y; w(m)) = G(x, y) + \sum_{s=1}^{m} (-4\pi\alpha/m)^s \sum\nolimits_{(s)} G(x, w_{i_1}) G(w_{i_1}, w_{i_2}) \cdots G(w_{i_{s-1}}, w_{i_s}) \times G(w_{i_s}, y).$$

Here the indices $(i_1, \cdots, i_s)$ in $\sum_{(s)}$ run over all $1 \leq i_1, \cdots, i_s \leq m$ satisfying $i_1 \neq i_2, i_2 \neq i_3, \cdots, i_{s-1} \neq i_s$. Let $\boldsymbol{G}_m$ (resp. $\boldsymbol{H}_m$) be the integral operator whose integral kernel function is $G_m(x, y; w(m))$ (resp. $h_m(x, y; w(m))$). Let $\|T\|_{2;m}$ denote the operator norm of a bounded linear operator T on the space of square integrable functions on $\Omega_{\alpha/m, w(m)}$. A key to our Theorem 1 is the following:

Proposition 1. *For a constant $C_{\varepsilon'}$ independent of $\{w(m)\}_{m=1}^{\infty} \in \mathcal{O}$,*

$$\|\boldsymbol{G}_m - \boldsymbol{H}_m\|_{2;m} \leq C_{\varepsilon'} m^{-1+\varepsilon'} q_{m,\alpha}$$

holds for any $\varepsilon' > 0$, where

$$q_{m,\alpha} = 1 + \sum_{s=1}^{m-1} (4\pi\alpha)^{s+1} \kappa^s + (4\pi\alpha\kappa)^m m.$$

Here

$$\kappa = \sup_m \left(m^{-1} \max_i \sum_{\substack{1 \leq r \leq m \\ r \neq i}} G(w_i, w_r) \right).$$

$\boldsymbol{H}_m$ converges to

$$\boldsymbol{G} + \sum_{s=1}^{\infty} (-4\pi\alpha)^s \boldsymbol{G}(V\boldsymbol{G})^s$$

when α is small, which is a left inverse of $-\Delta + 4\pi\alpha V$. Along this line, we get Theorem 1 when α is small. We need a slight modification of our proof for general α.

When $m=1$, h_m reduces to the integral kernel function $h_\alpha(x, y)$ on p. 771 of Ozawa [7]. By using this integral kernel function, we gave an asymptotic formula for eigenvalues of the Laplacian under singular variation of domains. For any $1 < m < \infty$, we can also prove the asymptotic formulas for eigenvalues by using h_m. Observing Theorem 1 we can say that h_m is a nice asymptotic Green's function for all $m = 1, 2, \cdots, \infty$.

We make a historical remark. By purely analytic method, Huruslov-Marchenko [4] studied various boundary value problems in a region with many small holes. See also Huruslov [3]. Their method is not perturbational and is potential theoretic. It seems to the author that our approach to "many small holes problems" based on perturbational calculation is new. There are papers concerning Theorem 1. Kac [5] treated eigenvalue problems in a region with many small holes in a probabilistic context. See also Rauch-Taylor [10], Papanicolaou-Varadhan [9] in which many interesting results were shown. See also Simon [11], Lions [6], Bensoussan-Lions-Papanicolaou [1] and Cioranescu [2].

Details of this paper will be given in Ozawa [8].

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