# 112. Cohomology Groups of the Unit Group of a Local Field

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1. Let $k$ be a local field, that is a complete field with respect to a discrete valuation. We assume that the residue class field of $k$ is finite. Let $K/k$ be a finite Galois extension of degree $n$ with the group $G$. We denote the unit group of $K$ by $U_K$. In this paper, we shall discuss relations between the cohomology groups $H^p(G, U_K)$ and the canonical class $\xi_{K,k}$ for $K/k$. For the notation and terminology, we use that given in S. Iyanaga [1].

There exists an exact sequence of $G$-modules ($U_K$ and $K^\times$ are written multiplicatively).

(1) $$1 \longrightarrow U_K \xrightarrow{\alpha} K^\times \xrightarrow{\beta} Z \longrightarrow 1.$$

Here $\beta$ is the normal exponential valuation with respect to $K$. Let us denote the inertia group, the ramification exponent and the relative degree of $K/k$ by $G_T$, $e$ and $m$, respectively. Then, from the cohomology sequences belonging to the exact sequence (1), we obtain $H^0(G, U_K) \cong G_T/[G, G]$, $H^1(G, U_K) \cong Z/eZ$. Then, by using the cup product, we have

$$\beta_*(H^2(G, K^\times)) \cong \beta_*(H^0(G, K^\times)) \cong Z/mZ.$$

Let us denote the lifting homomorphism from $G/G_T$ to $G$ by $\lambda$. Then it is easy to show that

$$\ker \delta_* = \beta_*(H^2(G, K^\times)) = \lambda(H^2(G/G_T, Z)) \cong Z/mZ.$$

Therefore, from the cohomology sequences belonging to (1), we have $H^2(G, U_K) \cong Z/eZ$, $H^3(G, U_K) \cong G_T/[G, G]$.

2. We shall show the existence of Artin's splitting group whose $p$-th cohomology group is isomorphic to $H^{p+1}(G, U_K)$. To define the splitting module, first we fix a 2-cocycle $f$ contained in the canonical class $\xi_{K,k}$. Let $\Pi_K$ be a prime element of $K$. Then, for every $\sigma$ and $\tau \in G$, $f[\sigma, \tau]$ is written in a form

$$f[\sigma, \tau] = \Pi_K^{\eta[\sigma,\tau]} u_{\sigma,\tau},$$

in which $u_{\sigma,\tau} \in U_K$ and $\eta[\sigma, \tau] \in Z$. For a cocycle $g$, we denote the cohomology class containing $g$ by $\{g\}$. Then we have $\beta_*(\xi_{K,k}) = \{\eta\}$. Let $\bar{Z}$ be Artin's splitting group of $\eta$. Then we have the following exact sequences of $G$-modules

(2) $$0 \longrightarrow Z_* \longrightarrow Z[G] \longrightarrow Z \longrightarrow 0,$$

$$(3) \qquad 0 \longrightarrow Z \xrightarrow{\varphi} \bar{Z} \xrightarrow{\psi} Z_* \longrightarrow 0.$$

Here $Z[G]$ is the group ring of $G$ and $Z_*$ is the free $Z$-module generated by $d_\sigma = \sigma - 1 (\sigma \neq 1, \sigma \in G)$. (See [1], Ch. 1, § 3.3.)

Our purpose is to show that the cohomology sequence derived from (1) is isomorphic to the cohomology sequence derived from (3). Let us denote by $\delta_*^{(i)}$ the connecting homomorphism belonging to the cohomology sequence derived from the exact sequence ($i$) ($1 \leqq i \leqq 3$). Then, for every $x \in H^p(G, Z)$, we have $\delta_*^{(3)} \circ \delta_*^{(2)}(x) = \beta_*(\xi_{K,k} \cup x)$, where $\cup$ denotes the cup product. We put

$$A_p = H^{p+1}(G, Z)/\beta_*(H^{p+1}(G, K^\times)) = H^{p+1}(G, Z)/\delta_*^{(3)}(H^p(G, Z_*)),$$
$$N_p = (\delta_*^{(2)})^{-1}(\ker \delta_*^{(3)}) \subset H^p(G, Z).$$

For our purpose, it is sufficient to show the existence of an isomorphism $\nu_p$ such that the following diagram is commutative

$$(4) \qquad \begin{array}{ccccccccc}
0 & \longrightarrow & A_p & \xrightarrow{\delta_*^{(1)}} & H^{p+2}(G, U_K) & \xrightarrow{\alpha_*} \ker \beta_* \hookrightarrow & N_p & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \nu_p} & & \| & & \\
0 & \longrightarrow & A_p & \xrightarrow{\varphi_*} & H^{p+1}(G, \bar{Z}) & \xrightarrow{\psi_*} \ker \delta_*^{(3)} \hookrightarrow & N_p & \longrightarrow & 0.
\end{array}$$

3. In this section, we shall define the isomorphism $\nu_p$. There exists a correspondence of cochains which may not be cochains denoted also by $\cup$, which induces the cup product. (For details, see [1], Ch. 1, § 6.4.)

Let $h$ be a cocycle of $N_p$ and $g : C_{p+1} \to Z$ be a $(p+1)$-cochain such that

$$\eta \cup h = \delta g \cdots (*).$$

Let us denote by $S_p$ the set consisting of all the pairs $(h, g)$ which satisfy the condition $(*)$. Then we can define an equivalence relation in $S_p$ as follows:

$(h, g)$ is equivalent to $(h', g')$ if there exist a $(p-1)$-cochain $k_1$, a $(p-1)$-cocycle $k_2$ and a $p$-cochain $k_3$ such that $h' = h + \delta k_1$, $g' = g + \eta \cup (k_1 + k_2) + \delta k_3$.

Let us denote by $\{(h, g)\}$ the equivalence class which contains $(h, g)$ and put $\tilde{S}_p = \{\{(h, g)\} \mid (h, g) \in S_p\}$. Since the above equivalence relation is compatible with the natural addition in $S_p$, $\tilde{S}_p$ naturally has the structure of an additive group. Then $\tilde{S}_p$ is an extension of $A_p$ by $N_p$ in the following sense: We have an exact sequence

$$0 \longrightarrow A_p \xrightarrow{a_p} \tilde{S}_p \xrightarrow{n_p} N_p \longrightarrow 0,$$

where $a_p$ and $n_p$ are determined respectively by $a_p(\{g\} \bmod \beta_*(H^{p+1}(G, K^\times))) = \{(0, -g)\}$ for $\{g\} \in H^{p+1}(G, Z)$ and $n_p(\{(h, g)\}) = \{h\}$ for $(h, g) \in S_p$. For a $(p+1)$-cochain $g : C_{p+1} \to Z$, there exists a $(p+1)$-cochain $\tilde{g} : C_{p+1}$

$\to K^\times$ such that $\beta\tilde{g}=g$. Then an onto homomorphism $\omega_p$ from $\tilde{S}_p$ to $H^{p+2}(G, U_K)$ is defined by putting $\omega_p((h, g))=\{(f \cup h)/\delta\tilde{g}\}$. We can easily verify that $\omega_p((h, g))=\omega_p((h', g'))$ if and only if $(h, g)$ is equivalent to $(h', g')$. Hence $\omega_p$ induces an isomorphism $v_p$ from $\tilde{S}_p$ to $H^{p+2}(G, U_K)$ such that $v_p(\{(h, g)\})=\{(f \cup h)/\delta\tilde{g}\}$.

**Proposition 1.** *For every $p \in \boldsymbol{Z}$, there exists an isomorphism $v_p$ such that the following diagram is commutative*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_p & \xrightarrow{a_p} & \tilde{S}_p & \xrightarrow{n_p} & N_p & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle v_p} & & \| & & \\ 0 & \longrightarrow & A_p & \longrightarrow & H^{p+2}(G, U_K) & \longrightarrow & N_p & \longrightarrow & 0. \end{array}$$

Next, we shall show that $\tilde{S}_p$ is also isomorphic to $H^{p+1}(G, \bar{\boldsymbol{Z}})$. We define an isomorphism $w_p$ from $\tilde{S}_p$ to $H^{p+1}(G, \bar{\boldsymbol{Z}})$ in each of the three cases i) $p \geqq 0$, ii) $p=-1$ and iii) $p \leqq -2$.

i) For every $p$-cocycle $h$ of $N_p$, we define a $(p+1)$-cochain $\bar{h}: C_{p+1} \to \bar{\boldsymbol{Z}}$ by

$$\bar{h}[\sigma_1, \cdots, \sigma_{p+1}]=h[\sigma_2, \cdots, \sigma_{p+1}]d_{\sigma_1} \qquad (\sigma_i \in G).$$

Then we have $\delta\bar{h}=\eta \cup h$. Therefore we can define an isomorphism $w_p$ by putting

$$w_p(\{(h, g)\})=\{\bar{h}-g\}.$$

ii) Since $H^{-1}(G, \boldsymbol{Z})=0$, we can take a $(-2)$-cochain $h_1$ such that $h=\delta h_1$. In this case, an isomorphism $w_p$ is defined by putting

$$w_p(\{(h, g)\})=\{\eta \cup h_1 - g\}.$$

iii) If $p<-2$, we define a $(p+1)$-cochain $\bar{h}$ for a $p$-cocycle $h$ of $N_p$ by

$$\bar{h}\langle\sigma_1, \cdots, \sigma_{q-1}\rangle=\sum_{\sigma \in G} h\langle\sigma^{-1}, \sigma_1, \cdots, \sigma_{q-1}\rangle d_\sigma \qquad (\sigma_i \in G),$$

where $q=-p-1$. In case $p=-2$ (i.e. $q=1$), $\bar{h}$ is defined by

$$\bar{h}\langle\ \rangle=\sum_{\sigma \in G} h\langle\sigma^{-1}\rangle d_\sigma.$$

Then, in both cases, $\bar{h}$ satisfies $\delta\bar{h}=-\eta' \cup h$. Here $\eta'$ denotes the 2-cocycle determined by $\eta'[\tau_1, \tau_2]=\eta[\tau_2^{-1}, \tau_1^{-1}]$ $(\tau_1, \tau_2 \in G)$. Let $k_0$ be a 1-cocycle determined by $k_0[\sigma]=\eta[\sigma, \sigma^{-1}]-\eta[\sigma, 1]$ $(\sigma \in G)$. Then $\eta'$ satisfies $\eta'+\eta=\delta k_0$. We define an isomorphism $w_p$ by

$$w_p(\{(h, g)\})=\{\bar{h}-g+k_0 \cup h\}.$$

Hence, similarly to Proposition 1, we obtain:

**Proposition 2.** *For every $p \in \boldsymbol{Z}$, there exists an isomorphism $w_p$ such that the following diagram is commutative*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_p & \xrightarrow{a_p} & \tilde{S}_p & \xrightarrow{n_p} & N_p & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle w_p} & & \| & & \\ 0 & \longrightarrow & A_p & \longrightarrow & H^{p+1}(G, \bar{\boldsymbol{Z}}) & \longrightarrow & N_p & \longrightarrow & 0. \end{array}$$

Let us denote $w_p \circ v_p^{-1}$ by $\nu_p$. Then we obtain the main theorem.

**Theorem.** *For every $p \in \boldsymbol{Z}$, we have an isomorphism*

$$\nu_p : H^{p+2}(G, U_K) \cong H^{p+1}(G, \bar{Z})$$

*such that the diagram* (4) *is commutative.*

When $K/k$ is totally ramified, we see that $\eta$ is cohomologous to 0 in $H^2(G, Z)$ because $\beta_*(H^2(G, K^\times)) = \lambda(H^2(G/G_T, Z)) = 0$ in this case. Therefore, from the above theorem, we have the following corollary.

**Corollary.** *If $K/k$ is totally ramified, we have*

$$H^{p+2}(G, U_K) \cong H^{p+1}(G, Z) \times H^p(G, Z) \qquad (p \in Z).$$

**Remark.** The $G$-module $\bar{Z}$ is uniquely determined by the cohomology class $\{\eta\}$ up to $G$-isomorphisms. Therefore we can say that the cohomology group $H^p(G, U_K)$ is completely described by the canonical class $\xi_{K,k}$. On the other hand, $\{\eta\}$ is characterized as one of the generators of the cyclic group $\lambda(H^2(G/G_T, Z))$. Hence, in the group theoretical aspects, we may say that $H^p(G, U_K)$ is described in terms of $G$ and $G_T$.

## Reference

[1] S. Iyanaga (ed.): The Theory of Numbers. North Holland/American Elsevier (1975).