107. Applications of the Multiplication of the Ito-Wiener Expansions to Limit Theorems

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(Communicated by Kôsaku YOSIDA, M. J. A., Nov. 12, 1982)

We are dealing with a real stationary process

$$X(t)=\sum_{k=1}^{\infty}\int c_k(\lambda)e_k(\lambda,t)d^k\beta, \qquad -\infty<t<\infty,$$
$$e_k(\lambda,t)=\exp(i[\lambda_1+\cdots+\lambda_k]t), \qquad d^k\beta=d\beta(\lambda_1)\cdots d\beta(\lambda_k),$$
$$\lambda=(\lambda_1,\cdots,\lambda_k), \qquad c_k \text{ are symmetric},$$
$$\bar{c}_k(\lambda)=c_k(-\lambda), \qquad c_k\in L^2(d^k\sigma=d\sigma(\lambda_1)\cdots d\sigma(\lambda_k)),$$

where $d\beta$ is the random spectral measure of a real Gaussian stationary process, with $E|d\beta|^2=d\sigma$, which is absolutely continuous $d\sigma(\lambda)=f(\lambda)d\lambda$. We exemplify the multiplication rule through the following simple case.

Let $f,g\in L^2(d^2\sigma)$, then

$$\int f(\lambda,\mu)d^2\beta\int g(\lambda,\mu)d^2\beta$$
$$=\int (f(\lambda,\mu)g(-\lambda,-\mu)+f(\lambda,\mu)g(-\mu,-\lambda))d^2\sigma$$
$$+\int d^2\beta\int \{f(\lambda,\lambda_1)g(-\lambda,\lambda_2)+f(\lambda,\lambda_1)g(\lambda_2,-\lambda)$$
$$+f(\lambda_1,\lambda)g(-\lambda,\lambda_2)+f(\lambda_1,\lambda)g(\lambda_2,-\lambda)\}d\sigma(\lambda)$$
$$+\int f(\lambda_1,\lambda_2)g(\lambda_3,\lambda_4)d^4\beta.$$

Define

$$\Phi(\zeta)=\sum_{m=0}^{\infty}\|c_m\|_2\zeta^m, \qquad \|c_m\|_2^2=\int |c_m|^2\,d^m\sigma,$$
$$M_{2m}=\{\xi\in L^2(\beta): |||\xi|||_{2m}<\infty\},$$

where

$$(|||\xi|||_{2m})^{2m}=\int_0^{\infty}d\mu_m(x)\frac{1}{2\pi}\int_0^{2\pi}|\Phi(\sqrt{mx}e^{i\varphi})|^{2m}\,d\varphi,$$
$$d\mu_m(x)=e^{-x}x^{m-1}dx/(m-1)!, \qquad 1\le m<\infty.$$

Theorem 1. *Suppose we are given $\xi_1,\cdots,\xi_m\in M_{2m}$ and let their IW-expansions be*

$$\xi_i=c_0^i+\sum_{k\ge 1}\int c_k^i(\lambda)d^k\beta, \qquad 1\le i\le m.$$

Multiply the right-hand sides term by term by the multiplication rule as above, and get a formal series of homogeneous polynomials, then *the series is unconditionally convergent in* $L^2(\beta)$, i.e. *it is convergent to the same limit in* $L^2(\beta)$, *regardless of the order of summation. Collect polynomials of the same degree into single terms and rearrange them in ascending degrees, then we get the IW-expansion of* $\xi_1\cdots\xi_m$.

Define

$$\Psi_k(h)=k!\int_{R^{k-1}} f(\lambda_1)\cdots f(\lambda_{k-1})d\lambda_1\cdots d\lambda_{k-1} \times\sup_x\int_x^{x+h}|c_k(\lambda,\lambda_1,\cdots,\lambda_{k-1})|^2 f(\lambda)d\lambda,$$

$$\Phi(|c_k|^2,h)=\int_0^h f_{2,k}(\lambda)d\lambda \qquad (1\le k<\infty),$$

where

$$f_{2,1}(\lambda)=c_1(\lambda)f(\lambda),$$

$$f_{2,k}(\lambda)=k!\int_{R^{k-1}}|c_k(\lambda-\lambda_1-\cdots-\lambda_{k-1},\lambda_1,\cdots,\lambda_{k-1})|^2 \times f(\lambda-\lambda_1-\cdots-\lambda_{k-1})\prod_{j=1}^{k-1}f(\lambda_j)d\lambda_j \qquad (k\ge 2).$$

Theorem 2. *Assume that*

(i) $\lim_{n\to\infty}\overline{\lim_{h\downarrow 0}}\frac{1}{h}\sum_{k\ge n}\Psi_k(h)=0$,

(ii) f *is bounded,*

(iii) $v(T)=E\left(\int_0^T X(t)dt\right)^2 \underset{\cap}{\cup} T \ (T\to\infty)$,

(iv) $\lim_{h\downarrow 0}\Phi(\delta|c_k|^2,h)/h=0$, $h=1/T$, *for every* $\varepsilon>0$ *and* k, *where* $\delta|c_k|^2=|c_k|^2-|c_k|^2\wedge(\varepsilon T^{1/3})$.

Then, as $T\to\infty$

$$v(T)^{-1/2}\int_0^T X(t)dt\to N(0,1) \qquad \text{(in distribution).}$$

Correspondingly to Theorem 1, we obtain a limit theorem for L^2-functionals built on the shifts of Brownian sheet process.

The subsequent application is to a generalization of Ibragimov's result [2] concerning the periodogram.

Define

$$\xi_T(\lambda)=\sqrt{T}\left(\int_0^\lambda I(x)dx-E\left(\int_0^\lambda I(x)dx\right)\right), \qquad 0\le\lambda\le\infty,$$

$$I(x)=\frac{1}{2\pi T}\left|\int_0^T X(t)e^{-ixt}dt\right|^2,$$

and let $\xi_\infty(\lambda)$, $0\le\lambda\le\infty$, be the Gaussian process with

$$E(\xi_\infty(\lambda))=0.$$

$$\mathrm{Cov}\,(\xi_\infty(\lambda),\xi_\infty(\mu))=2\pi\left(\int_0^\lambda\int_0^\mu f_4(\alpha,-\alpha,\beta)\,d\alpha d\beta+\int_0^{\lambda\wedge\mu}f_2^2(\alpha)\,d\alpha\right),$$

where f_m denotes the m-th cumulant spectral density of $X(t)$.

Theorem 3. *Suppose that*

(A) $c_k(k\ge 2)$ *are bounded Borel functions,*

(B) $f, fc_1^2 \in L^2$,

(C) (i) *for any* $a>0$, $n\geq 2$, *and* k *with* $0\leq k\leq n-2$

$$\lim_{\varepsilon\to 0}\int_{|x_j|\leq a}\cdots\int |c_n(x_1,\cdots,x_{n-1},\varepsilon-x_1-\cdots-x_k) - c_n(x_1,\cdots,x_{n-1},-x_1,\cdots,-x_k)|\,dx_1\cdots dx_{n-1}=0,$$

(ii) $\int_0^h |f(x)-f(0)|\,dx=o(h),\ h\to +0,$

(D) $\sum_{k=0}^{\infty} k!\,3^k(b*b)_k^2<\infty,\ b=(b_0, b_1, \cdots),\ b_0=b_1=0,$

$$b_k=\|c_k\|_\infty \|f\|_1^{k/2}\ (k\geq 2),$$

$\|\cdot\|_1=L^1$*-norm*, $*=$*convolution*.

Then, as $T\to\infty$, *every finite-dimensional distribution of* ξ_T *converges weakly to the corresponding one of* ξ_∞.

Theorem 4. *Suppose that* $X(t)$ *satisfies the conditions in Theorem 3 except* (D) *and assume further that*

(i) f *is bounded and one can find* ε, $0<\varepsilon<1$, *such that*

$$\int_0^\infty \lambda^\varepsilon f(\lambda)d\lambda<\infty,$$

(ii) $\sum_{n=0}^{\infty} n!\,7^n(b^{4*})_n^2<\infty.$

Then, when $T\to\infty$, *as* $C[0,\infty]$*-valued random variables,* ξ_T *converges in distribution to* ξ_∞.

References

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