105. On Surfaces of Class VII_0 with Curves

By Iku NAKAMURA
Department of Mathematics, Hokkaido University

(Communicated by Kunihiko KODAIRA, M. J. A., Nov. 12, 1982)

Introduction. A minimal compact complex surface is called a surface of class VII_0 or in short a VII_0 surface if b_1 (the first Betti number) is equal to 1. The purpose of this note is to announce recent results on the classification of VII_0 surfaces with curves, based on Enoki's and my articles [1], [7]. Most of the results will be summarized in a table (4.9).

Notations. We denote by S a compact complex surface, by $\mathcal{O}_S$ the sheaf of holomorphic functions on S, by b_i the i-th Betti number of S. For two effective divisors C and C' we denote by CC' the intersection number of C and C', by C^2 the selfintersection number of C, by $\mathcal{O}_C$ the structure sheaf of C, that is $\mathcal{O}_S/\mathcal{O}_S(-C)$, by $h^1(C, \mathcal{O}_C)$ $\dim_C H^1(C, \mathcal{O}_C)$, by $b_2(C)$ the number of irreducible components of C. We also denote by K_S the canonical bundle of S.

§1. Elliptic VII_0 surfaces. In view of the theorem of Chow and Kodaira a compact complex surface with two algebraically independent meromorphic functions is a projective surface, hence b_1 is even. Therefore no surface of class VII_0 has two algebraically independent meromorphic functions. First we recall

(1.1) **Theorem [6].** *If a VII_0 surface S has a nonconstant meromorphic function, then S is an elliptic surface and isomorphic to one of the following surfaces:*

$$L_{a_r}(m_r, \beta_r)L_{a_{r-1}}(m_{r-1}, \beta_{r-1})\cdots L_{a_1}(m_1, \beta_1)(\boldsymbol{P}^1 \times C)$$

where $\boldsymbol{P}^1$ is a rational curve, C is an elliptic curve, $a_k \in \boldsymbol{P}^1$, $a_j \neq a_k$ $(j \neq k)$, m_k is a positive integer, β_k is a point of C of order m_k and $\beta_1+\beta_2+\cdots+\beta_k \neq 0$.

(1.2) In view of (1.1) the problem of classifying VII_0 surfaces is reduced to that of classifying VII_0 surfaces with no meromorphic functions except constants. We shall discuss exclusively VII_0 surfaces with curves in this note. We notice that there are VII_0 surfaces with no curves and it remains unsettled to classify all such surfaces. See [2].

§2. Curves on VII_0 surfaces.

(2.1) **Lemma [7].** *Let S be a VII_0 surface with no meromorphic functions except constants, D an effective divisor on S. Then*

1) $h^1(D, \mathcal{O}_D) \leqq 2$,

2) *if D is reduced and connected, then $h^1(D, \mathcal{O}_D)\leqq 1$,*
3) *if D is irreducible, then D is either a nonsingular rational curve, or a rational curve with a node, or a nonsingular elliptic curve,*
4) *if D is reduced and connected, and if $h^1(D, \mathcal{O}_D)=1$, $h^1(E, \mathcal{O}_E)=0$ for any proper subcurve E of D, then D is either a nonsingular elliptic curve, or a cycle of rational curves.*

(2.2) In (2.1) I mean by a cycle of rational curves a reduced curve $C=\sum_{\nu=1}^{n} C_\nu$ such that $n\geqq 3$, C_ν is a nonsingular rational curve, $C_\nu C_{\nu+1}=1$, $C_\nu C_\mu=0$ ($\nu\neq\mu$, $\mu\pm 1 \bmod n$) or that $n=2$, C_ν is a nonsingular rational curve, C_1 and C_2 meet transversally at two points, or that $n=1$, C_1 is a rational curve with a node.

(2.3) **Lemma** [7]. *Let S be a VII_0 surface with no meromorphic functions except constants, D a reduced divisor on S. Suppose that $h^1(D, \mathcal{O}_D)=2$, $h^1(E, \mathcal{O}_E)\leqq 1$ for any proper subcurve E of D. Then $K_S+D=0$, and*
1) *$D=E_1+E_2$, E_ν a nonsingular elliptic curve with $E_\nu^2=0$, or*
2) *$D=E+Z$, E a nonsingular elliptic curve, Z a cycle of rational curves with $E^2<0$, $Z^2=0$, or*
3) *$D=A+B$, A and B cycles of rational curves with $A^2<0$, $B^2<0$.*

(2.4) We notice that a VII_0 surface with a cycle of rational curves has no meromorphic functions except constants by (1.1).

§ 3. Inoue surfaces and exceptional compactifications.

(3.1) Let M be a complete module in a real quadratic field K, $U(M)=\{\alpha\in K\,;\alpha M=M\}$, $U^+(M)=\{\alpha\in K\,;\alpha M=M,\ \alpha>0,\ \alpha'>0\}$, V a subgroup of $U^+(M)$ of finite index. Then M and V act on the product $\boldsymbol{H}\times\boldsymbol{C}$ of the upper half plane and the complex plane by

$$\alpha:(z_1, z_2)\longrightarrow(\alpha z_1, \alpha' z_2),\qquad m:(z_1, z_2)\longrightarrow(z_1+m,\ z_2+m').$$

Let $G(M, V)$ be the group generated by the above actions of M and V. Then $G(M, V)$ acts upon $\boldsymbol{H}\times\boldsymbol{C}$ freely and properly discontinuously so that we have a complex surface $S'(M, V)$ with no singularities as quotient. This $S'(M, V)$ is compactified into a VII_0 surface $S(M, V)$ by adding two suitable cycles A and B of rational curves [4]. We call $S(M, V)$ a hyperbolic Inoue surface.

(3.2) In general $[U(M):U^+(M)]=1$ or 2. When $[U(M):U^+(M)]=2$, we choose a subgroup V of $U(M)$ of odd index. Let $V^2=\{\alpha^2\,;\alpha\in V\}$. Then V^2 is a subgroup of $U^+(M)$, and the group V/V^2 of order two acts on $S(M, V^2)$ freely so that we have a VII_0 surface $S(M/V^2)/(V/V^2)$ as quotient which we denote by $\hat{S}(M, V)$ and call a half Inoue surface.

(3.3) There is another series of Inoue surfaces which were given in [3]. This series of surfaces are denoted by $S(t, n)$ where $t\in\boldsymbol{C}$, $0<|t|<1$, n is a positive integer and we call them parabolic Inoue surfaces. Any parabolic Inoue surface $S(t, n)$ has an elliptic curve E and a cycle Z of n rational curves with $E^2=-n$, $Z^2=0$. It is a compactification of a line bundle of degree $-n$ over the elliptic curve E by the cycle Z.

(3.4) Let A be an affine line bundle over an elliptic curve E. Let L be the linear part of A. Then L is a line bundle over E. Suppose $\deg L=-n<0$. Then A has an elliptic ruled surface as a natural compactification by the elliptic curve E. A has another "exceptional" compactification $S(A)$ by a cycle C of n rational curves with $C^2=0$ which is a VII_0 surface. We call this $S(A)$ an exceptional compactification of A of degree n. $S(A)$ is a parabolic Inoue surface if and only if $S(A)$ has an elliptic curve. See [1].

§ 4. Characterizations and a table.

(4.1) **Theorem** (Kato, see [7]). *Let S be a VII_0 surface with no meromorphic functions except constants. Suppose S has two elliptic curves. Then S is a primary Hopf surface.*

(4.2) **Theorem** [7]. *Let S be a VII_0 surface with no meromorphic functions except constants. Suppose S has an elliptic curve but no cycle of rational curves. Then S is a primary Hopf surface.*

See [6] for the definition of primary Hopf surfaces.

(4.3) **Theorem** [1]. *Let S be a VII_0 surface. Suppose S has a cycle C of rational curves with $C^2=0$. Then S is an exceptional compactification $S(A)$ of an affine line bundle A over an elliptic curve.*

(4.4) **Theorem** [1], [7]. *Let S be a VII_0 surface. Suppose S has an elliptic curve and a cycle of rational curves. Then S is a parabolic Inoue surface.*

This theorem was first proved by Enoki as a special case of (4.3) by applying (2.3). A more direct proof was given in [7].

(4.5) **Theorem** [7]. *Let S be a VII_0 surface. Suppose S has two cycles of rational curves. Then S is a hyperbolic Inoue surface.*

(4.6) **Theorem** [7]. *Let S be a VII_0 surface with C a cycle of rational curves with $C^2<0$. Suppose that S satisfies one of the following equivalent conditions.*

1) *There exists a flat line bundle F such that $K_S+C=F$.*

2) *$b_2=\sharp$ (irreducible components of C).*

3) *$C^2=-b_2$.*

4) *The natural homomorphism i_* of $H_1(C,Z)$ to $H_1(S,Z)$ is not surjective.*

5) *$[H_1(S,Z):i_*H_1(C,Z)]=2$.*

Then S is a half Inoue surface.

(4.7) **Theorem** [7]. *Let S be a VII_0 surface. Suppose S has an elliptic curve. Then S is one of the following surfaces;*

elliptic VII_0 *surfaces* (1.1) ($b_2=0$),

primary Hopf surfaces ($b_2=0$),
parabolic Inoue surfaces ($b_2>0$).

(4.8) **Theorem** [7]. *Let S be a* VII$_0$ *surface with* $b_2=1$ *having at least a curve. Then S is either an exceptional compactification of degree one or a half Inoue surface* $\hat{S}(M, U(M))$, $M=\boldsymbol{Z}+\boldsymbol{Z}(3+\sqrt{5})/2$.

Now we have the following classification table of VII$_0$ surfaces.

(4.9) **Table.** [1]+[6]+[7]

curves	surfaces
1) (more than) 3 elliptic curves	elliptic VII$_0$ surfaces
2) 2 elliptic curves	primary Hopf surfaces
3) an elliptic curve and no cycle	primary Hopf surfaces
4) an elliptic curve and a cycle	parabolic Inoue surfaces
5) two cycles	hyperbolic Inoue surfaces
6) a cycle C with $C^2=0$	exceptional compactifications with no elliptic curves
7) a cycle C with $C^2<0$	
7-1) $b_2(S)=b_2(C)$	half Inoue surfaces
7-2) $b_2(S)>b_2(C)$	? ? ?

(4.10) We notice that there are a lot of VII$_0$ surfaces with cycles C with $C^2<0$ and $b_2(S)>b_2(C)$. See [5], [8]. We also notice that the converse of (4.3)–(4.8) are true.

References

[1] I. Enoki: Surfaces of class VII$_0$ with curves. Tôhoku Math. J., **33**, 453–492 (1981).
[2] M. Inoue: On surfaces of class VII$_0$. Invent. math., **24**, 269–310 (1974).
[3] ——: New surfaces with no meromorphic function. Proc. Int. Cong. of Math., Vancouver, vol. 1, pp. 423–426 (1974).
[4] ——: ditto. II. Complex Analysis and Algebraic Geometry. Iwanami Shoten Publ. and Cambridge Univ. Press, pp. 91–106 (1977).
[5] Ma. Kato: Compact complex manifolds containing global spherical shells I. Proc. Int. Symp. Algebraic Geometry, Kyoto, pp. 45–84 (1977).
[6] K. Kodaira: On the structure of compact complex analytic surfaces, II. Amer. J. Math., **88**, 682–721 (1966).
[7] I. Nakamura: On surfaces of class VII$_0$ with curves (preprint).
[8] ——: Rational degeneration of VII$_0$ surfaces (preprint).