10. *On the Inducing of Unipotent Classes for Semisimple Algebraic Groups.* I

Case of Exceptional Type

By Takeshi HIRAI
Department of Mathematics, Kyoto University

(Communicated by Shokichi IYANAGA, M. J. A., Jan. 12, 1982)

Let G be a connected semisimple algebraic group over an algebraically closed field K. Let P be a parabolic subgroup of G, L a Levi subgroup and U_P the unipotent radical of P. For a conjugate class C of unipotent elements in L, the induced class $C'=\mathrm{Ind}_{L,P}^{G}\, C$ is defined as the unique class in G which intersects CU_P densely [6]. General properties of this inducing were given in [6] (see also [1]), and in particular the induced class depends only on L and not on P. (So this class is also denoted as $\mathrm{Ind}_L^G\, C$.) Richardson classes of G are the special induced classes for which C's consist of unit element only, and we have studied them in detail in [5]. In this paper, we assume that the characteristic of K is zero and study how we can determine C' from C explicitly, when G is of exceptional type. The case of classical type will be treated in the succeeding paper. Using the results in this paper, we can make up easily the complete table of (C, C') by simple and formal processes.

We call a unipotent class *fundamental* if it can not be induced from any C of such a proper subgroup L. We first list up all the fundamental classes for every exceptional simple group G. Secondly we prove that, using the properties of the inducing given in [6] and the results in [5] on Richardson classes, we can reduce the explicit determination of C' to nine cases of fundamental C's. We note that these fundamental cases can be treated by the same idea as that in [5, § 4] for Richardson classes.

§ 1. Fundamental unipotent classes. Let G be simple. We list up all non-trivial fundamental classes by means of their (weighted) Dynkin diagrams, and also for each class the type of a "minimal containing regular subalgebra" of minimal rank in [2, Tables 17–20, pp. 177–185] (cf. also [7] and the corrected tables in [3]). The numbering of simple roots are as in [5].

Type G_2 10, 01. $(A_1, \tilde{A}_1)$

Type F_4 1000, 0001, 0100, 0010, 0101.
$(A_1,\ \tilde{A}_1,\ A_1+\tilde{A}_1$ (cf. [3]), $A_2+\tilde{A}_1,\ \tilde{A}_2+A_1)$

Type E_6 $\quad \underset{1}{00000}$, $\underset{0}{00100}$, $\underset{0}{10101}$. $(A_1,\ 3A_1,\ 2A_2+A_1)$

Type E_7 $\quad \underset{0}{000001}$, $\underset{0}{010000}$, $\underset{0}{000010}$, $\underset{1}{100000}$, $\underset{0}{000100}$, $\underset{0}{010010}$, $\underset{0}{000101}$.

$(A_1,\ 2A_1,\ [3A_1]',\ [4A_1]'',\ A_2+2A_1,\ 2A_2+A_1,\ [A_3+A_1]')$

Type E_8 $\quad \underset{0}{1000000}$, $\underset{0}{0000001}$, $\underset{0}{0100000}$, $\underset{1}{0000000}$, $\underset{0}{1000001}$, $\underset{0}{0010000}$, $\underset{0}{0000010}$, $\underset{0}{0100001}$,

$\underset{0}{0001000}$, $\underset{0}{1010000}$, $\underset{0}{1000010}$, $\underset{1}{0100000}$, $\underset{0}{0000100}$, $\underset{0}{0001001}$, $\underset{0}{0100100}$, $\underset{0}{1010010}$.

$(A_1,\ 2A_1,\ 3A_1,\ [4A_1]',\ A_2+A_1,\ A_2+2A_1,\ A_2+3A_1,\ 2A_2+A_1,$
$2A_2+2A_1,\ A_3+A_1,\ [A_3+2A_1]'$ (cf. [3]), $D_4(a_1)+A_1,$
$A_3+A_2+A_1,\ [2A_3]',\ A_4+A_3,\ D_5(a_1)+A_2).$

§2. **Reduction to fundamental cases.** For the determination of $C'=\mathrm{Ind}_{L,P}^G C$ from C, the following result is given in [6].

(I) Let Π be the set of simple roots of G. Assume that P is standard and let $\Pi_L \subset \Pi$ be the set of simple roots of L. The Dynkin diagram of C is given by putting for every $\alpha \in \Pi_L$ its weight 0, 1 or 2. Let us give weight 2 for every $\alpha \in \Pi-\Pi_L$. If thus obtained diagram is the Dynkin diagram of a class C'' of G, then $C'=C''$.

We can also prove the following.

(II) Let C be a Richardson class of L corresponding to $\Gamma \subset \Pi_L$ as in [5, §1]. Then C' is also a Richardson class of G corresponding to the same $\Gamma \subset \Pi$.

On the other hand, the Richardson class is determined from Γ by the result in [5]. Thus the induced class C' is determined by (II) by means of a formal manipulation of the equivalence relation $\sim$ in the set of Γ's in [5].

Example 1. $\mathrm{Ind}_{C_3}^{F_4} \cdot 010 = 0200$. (The dot "$\cdot$" indicates the position of an element in $\Pi-\Pi_L$.) In fact, the class 010 in C_3 is a Richardson class corresponding to $\Gamma_L = \bullet\Leftarrow\!\Rightarrow\bullet\!-\!\circ$ (the elements of $\Gamma_L \subset \Pi_L$ are indicated by black points). Therefore by (II) the induced class C' is a Richardson class corresponding to $\Gamma = \circ\!-\!\bullet\Rightarrow\bullet\!-\!\circ$. On the other hand, by (6) in [5, §2], $\Gamma \sim \Gamma' = \bullet\!-\!\circ\Rightarrow\bullet\!-\!\bullet$, and there exists a class with Dynkin diagram 0200. Hence the result.

Moreover the inducing has the following general property.

Lemma. *Let $Q \supset P$ be another parabolic subgroup of G with a Levi subgroup M. Then $M \supset L$, and* $\mathrm{Ind}_L^G C = \mathrm{Ind}_M^G \mathrm{Ind}_L^M C$.

As consequences of this lemma, we have the following.

(III) If we can find an intermediate M such that $C_1 = \mathrm{Ind}_L^M C$ and $\mathrm{Ind}_M^G C_1$ are given by a certain method, then $\mathrm{Ind}_L^G C$ is known.

(IV) Assume that $\mathrm{Ind}_L^G C$ is known, and put $C_1 = \mathrm{Ind}_L^M C$ for an intermediate M. Then $\mathrm{Ind}_M^G C_1$ is known.

By (IV) we know in particular that it is sufficient for us to determine $\mathrm{Ind}_L^G C$ for any proper $L \subset G$ and any fundamental class C in L. We use also the following.

(V) Let τ be an automorphism of G leaving stable every unipotent class in G. Then, $\mathrm{Ind}_L^G C=\mathrm{Ind}_{\tau(L)}^G \tau(C)$, where $\tau(C)=\{\tau(g): g\in C\}$.

Thus we can reduce the determination of $C'=\mathrm{Ind}_L^G C$ to a certain number of fundamental cases modulo simple and formal manipulations according to (I)–(V).

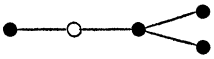
Note. For type F_4, we have L of type B_2, B_3 or C_3. For types E_6, E_7 and E_8, we also have type D_n $(4\leqslant n\leqslant 7)$ as simple components of L. Therefore we must have complete result for these classical groups. This can be given easily (see e.g. [4]). For example, fundamental classes are given as follows (by means of Dynkin diagrams).

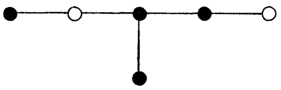
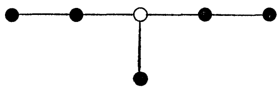
01 (in B_2); 010, (in B_3); 100 (in C_3); 01^{0}_{0}, 10^{1}_{1} (in D_4);

010^{0}_{0}, 101^{0}_{0} (in D_5); 0100^{0}_{0}, 0001^{0}_{0}, 1010^{0}_{0}, 1000^{1}_{1} (in D_6);

01000^{0}_{0}, 00010^{0}_{0}, 10100^{0}_{0}, 10001^{0}_{0} (in D_7).

Example 2. $\mathrm{Ind}_{B_2}^{F_4}\cdot 01\cdot=0202$. In fact, we know $\mathrm{Ind}_{B_2}^{B_3}\cdot 01=020$. Then the result is obtained by (III) and (I).

Example 3. $\mathrm{Ind}_{D_4}^{E_6}\cdot 0\underset{0}{1}0\cdot=00\underset{0}{2}00$. In fact, $\mathrm{Ind}_{D_4}^{D_5}\cdot 0\underset{0}{1}0=02\underset{0}{0}0$ is a Richardson class corresponding to $\Gamma_L=$ . Hence by (II) the induced class is a Richardson class corresponding to

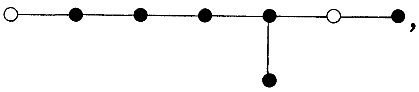
$\Gamma=$ , and then $\Gamma\sim\Gamma'=$ 

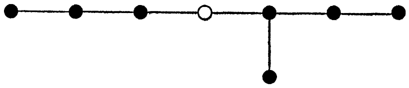
by (4) and (9) in [5, § 2]. Thus the result is obtained by (III) and (I).

Example 4. $\mathrm{Ind}_{A_2+D_4}^{E_8} 00\cdot 0\underset{0}{1}0\cdot=2000\underset{0}{2}00$. In fact, $\mathrm{Ind}_{D_4}^{D_5} 0\underset{0}{1}0\cdot=00\underset{0}{2}0$, and the argument is similar as in Example 3.

Example 5. $\mathrm{Ind}_{A_1+D_5}^{E_8}\cdot 0\cdot 0\underset{0}{1}01=2002\underset{0}{0}02$. In fact, the left hand side equals $\mathrm{Ind}_{A_1+D_5}^{E_8} 0\cdot 1\underset{0}{0}10\cdot$ by (V), and then $\mathrm{Ind}_{A_1+D_5}^{D_7} 0\cdot 1\underset{0}{0}10=200\underset{0}{2}00$. Hence the result by (I).

Example 6. $\mathrm{Ind}_{D_6}^{E_8}\cdot 000\underset{0}{1}0\cdot=0002\underset{0}{0}00$. In fact, $\mathrm{Ind}_{D_6}^{E_7} 000\underset{0}{1}0\cdot=0000\underset{0}{2}0$, as we will see in the next section. Hence the induced class is a Richardson class corresponding to

$\Gamma=$ , and then

$\Gamma\sim\Gamma'=$ 

by (5) and (9) in [5, § 2]. Hence the result.

§ 3. **Fundamental cases.** Using the results for B_2, B_3, C_3 and D_n $(4\leqslant n\leqslant 7)$, and those in [5], we reduce, by the process in § 2, all the cases to the following.

For type F_4, $\mathrm{Ind}_{B_3}^{F_4}\,010\cdot=1010$.

For type E_6, $\mathrm{Ind}_{D_5}^{E_6}\,\underset{0}{0100\cdot}=\underset{1}{10001}$; $\mathrm{Ind}_{D_5}^{E_6}\,\underset{0}{1010\cdot}=\underset{1}{01010}$.

For type E_7,

$\mathrm{Ind}_{E_6}^{E_7}\,\underset{1}{\cdot 00000}=\underset{0}{010001}$; (1) $\mathrm{Ind}_{E_6}^{E_7}\,\underset{0}{\cdot 00100}=\underset{0}{101001}$; $\mathrm{Ind}_{D_6}^{E_7}\,\underset{0}{10100\cdot}=\underset{0}{010100}$;

(2) $\mathrm{Ind}_{D_6}^{E_7}\,\underset{0}{00010\cdot}=\underset{0}{000020}$; $\mathrm{Ind}_{A_1+D_5}^{E_7}\,\underset{0}{0\cdot 0010}=\underset{0}{010100}$.

For type E_8,

$\mathrm{Ind}_{E_7}^{E_8}\,\underset{0}{\cdot 010000}=\underset{0}{0200000}$; $\mathrm{Ind}_{E_7}^{E_8}\,\underset{0}{\cdot 000010}=\underset{0}{0010001}$; (3) $\mathrm{Ind}_{E_7}^{E_8}\,\underset{0}{\cdot 010010}=\underset{0}{0101001}$;

$\mathrm{Ind}_{E_7}^{E_8}\,\underset{0}{\cdot 000101}=\underset{0}{0010100}$; $\mathrm{Ind}_{D_7}^{E_8}\,\underset{0}{010000\cdot}=\underset{0}{0010001}$; (4) $\mathrm{Ind}_{D_7}^{E_8}\,\underset{0}{100010\cdot}=\underset{1}{0100010}$;

$\mathrm{Ind}_{D_7}^{E_8}\,\underset{0}{000100\cdot}=\underset{0}{1000100}$; (5) $\mathrm{Ind}_{D_7}^{E_8}\,\underset{0}{001010\cdot}=\underset{0}{2002000}$;

(6) $\mathrm{Ind}_{A_1+E_6}^{E_8}\,\underset{1}{0\cdot 00000}=\underset{0}{1010001}$; $\mathrm{Ind}_{A_1+E_6}^{E_8}\,\underset{0}{0\cdot 00100}=\underset{0}{0010100}$;

(7) $\mathrm{Ind}_{A_1+E_6}^{E_8}\,\underset{0}{0\cdot 10101}=\underset{0}{2000200}$; (8) $\mathrm{Ind}_{A_2+D_5}^{E_8}\,\underset{0}{00\cdot 0101}=\underset{0}{0110101}$;

(9) $\mathrm{Ind}_{A_2+D_5}^{E_8}\,\underset{0}{00\cdot 0010}=\underset{0}{0002000}$.

In addition to (I)–(V) in § 2, we have one more powerful machinary (VI) to determine $C'=\mathrm{Ind}_L^G C$ from C.

(VI) $\dim C'=\dim C+\dim G/L$ [6, Th. 1.3 (a)].

Note that the dimension of every unipotent class is calculated from its Dynkin diagram.

Checking case by case, we see that, in the list above, the unnumbered cases can be determined by (VI). Hence the numbered cases (1)–(9) are left to prove. Moreover these cases can be treated as in [5, §§ 3–4].

Note. It is worth mentioning here the following for classical groups.

$\mathrm{Ind}_{A_1+D_4}^{D_6}\,0\cdot 10^{1}_{1}=2002^{0}_{0}$; $\mathrm{Ind}_{D_6}^{D_7}\,\cdot 0100^{0}_{0}=02000^{0}_{0}$; $\mathrm{Ind}_{D_6}^{D_7}\,\cdot 0001^{0}_{0}=01010^{0}_{0}$;

$\mathrm{Ind}_{A_2+D_4}^{D_7}\,00\cdot 01^{0}_{0}=01101^{0}_{0}$; $\mathrm{Ind}_{A_2+D_4}^{D_7}\,00\cdot 10^{1}_{1}=10110^{1}_{1}$.

The author expresses his hearty thanks to Profs. R. Hotta and T. Tanisaki who gave valuable comments and pointed out to him a mistake in his table of (C, C').

References

[1] W. Borho: Abh. Math. Sem. Univ. Hamburg, **51**, 1–4 (1981).

[2] E. B. Dynkin: Amer. Math. Transl., Ser. 2, **6**, 111–245 (1975); Mat. Sbornik, N. S., **30**, 349–462 (1952).

[3] A. G. Elašbili: Trudy Tbilisskovo Mat. Instituta, **46**, 109–132 (1975) (in Russian).

[4] W. H. Hesselink: Math. Z., **160**, 217–234 (1978).

[5] T. Hirai: Proc. Japan Acad., **57A**, 367–372 (1981).

[6] G. Lusztig and N. Spaltenstein: J. London Math. Soc., **19**, 41–52 (1979).

[7] K. Mizuno: Tokyo J. Math., **3**, 391–460 (1980).