75. Disjointness of Sequences $[\alpha_i n+\beta_i]$, $i=1, 2$

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1. Introduction. We shall give in this note a criterion for disjointness of two sequences $[\alpha_1 n+\beta_1]$ and $[\alpha_2 n+\beta_2]$ $(\alpha_1, \alpha_2>0)$ where $[x]$ denotes the greatest integer $\leqq x$, and n runs through the set N of positive integers. Such criterion is known if either α_1 or α_2 is irrational (cf. [1]). But in case α_1 and α_2 are both rational numbers, complete answer has not yet been known, although there are some investigations (cf. [1], [2]).

In the following, $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ have the usual meanings. (a, b) means the greatest divisor of a and b.

It is easy to see the following two facts: (1) If $\alpha=q/a$ where q and $a\in N$ and $(q, a)=1$, the effect of $\beta\in \boldsymbol{R}$ on the sequence $[\alpha n+\beta]$ depends only on $[a\beta]$. Hence, without changing the sequence $[\alpha n+\beta]$, β can be replaced by a rational number b/a whose denominator is a. (2) If both α_1 and $\alpha_2\in \boldsymbol{Q}$, two sequences $[\alpha_i n+\beta_i]$ $(i=1,2)$, $n\in N$ are disjoint if and only if two sets $\{[\alpha_i n+\beta_i] : n\in \boldsymbol{Z}\}$ $(i=1,2)$ are disjoint.

So let us consider now the sets $\{[(q_i n+b_i)/a_i] : n\in \boldsymbol{Z}\}$ $(i=1,2)$ with $(q_1, a_1)=(q_2, a_2)=1$, which we shall denote with $S(q_i, a_i, b_i)$ $(i=1,2)$. We put furthermore $(q_1, q_2)=q$, $(a_1, a_2)=a$, $a_i=au_i$ $(i=1,2)$.

Then we have

Theorem 1. *Notations being as above, consider q_1, q_2, a_1 and a_2 as given. Two sets $S(q_1, a_1, b_1)$ and $S(q_2, a_2, b_2)$ are disjoint with suitable two integers b_1 and b_2 if and only if*

$$xu_1+yu_2=q-2u_1u_2(a-1) \tag{1}$$

holds with some $(x, y)\in N\times N$.

In case this condition is satisfied, we can take a solution (x_0, y_0) of (1) such that $1\leqq y_0\leqq u_1$. Furthermore if $x_0>u_2$, define the numbers x_1 and y_1 by $x_1=x_0-u_2$ and $y_1=u_1-y_0$.

Theorem 2. *Assume that (q_i, a_i) $(i=1,2)$ satisfy the condition of Theorem 1. Then $S(q_i, a_i, b_i)$ $(i=1,2)$ are disjoint if and only if*

$$u_1b_2-u_2b_1\in(E_1\cup E_2) \pmod q,$$

where $E_1=\{u_1X+u_2Y+u_1u_2(a-1) : 0\leqq X\leqq x_0-1,\ 1\leqq Y\leqq y_0\}$ and $E_2=\{u_1X+u_2Y+u_1u_2(a-1) : 0\leqq X\leqq x_1-1,\ y_0+1\leqq Y\leqq u_1\}$. (In case $x_0\leqq u_2$, we define $E_2=\phi$.)

In the following we shall sketch the proof of Theorems 1 and 2. Details will appear elsewhere. Our results can be applied to the theory of "eventually covering families" (cf. [3]).

2. Notations and definitions. (i) If $f \in \boldsymbol{Z}$ and $h \in \boldsymbol{N}$ or 0, we write $[f, f+h]=\{f, f+1, \cdots, f+h\}$. This set is called a *segment* of $\boldsymbol{Z}$ *of length* $h+1$.

(ii) For $t \in \boldsymbol{Z}$ and $x \in \boldsymbol{Z}$, we denote $T_t\langle x\rangle = t+x$. This operation is applied also to any subset of $\boldsymbol{Z}$.

(iii) Let (q_i, a_i) $(i=1,2)$ be given as in the introduction, which we consider as fixed. Consequently, $q=(q_1, q_2)$ is also considered as given. ρ denotes the canonical map $\rho: \boldsymbol{Z} \to \boldsymbol{Z}/(q)$, and σ the map from $\boldsymbol{Z}$ to $\boldsymbol{C}$ defined by $\sigma(r)=\exp(2\pi i r/q)$ for $r \in \boldsymbol{Z}$. We put $C(q)=\sigma(\boldsymbol{Z})$. This is the set of q roots of unit. The σ image of a segment of $\boldsymbol{Z}$ is called a *segment of* $C(q)$, the *length* of which is defined as its cardinality $(\leqq q)$.

3. Sketch of the proof of Theorems 1 and 2. Let q_i, a_i, b_i, u_i $(i=1,2)$, q and a be as in the introduction. Besides these, we fix the number b_1 to be -1, and investigate a condition for b_2 such that $S(q_1, a_1, -1) \cap S(q_2, a_2, b_2)=\phi$.

Put $A=S(q_1, a_1, -1)$ and $B=S(q_2, a_2, b_2)$. Now we divide A into u_1 subsets as follows. Take $c \in \boldsymbol{Z}$ such that $cq_1 \equiv -1 \pmod{u_1}$. Put $A_j = S(q_1u_1, a_1, q_1cj-1)$ $(0 \leqq j \leqq u_1-1)$. Then $A=\bigcup_{j=0}^{u_1-1} A_j$ (disjoint union). We put $\boldsymbol{b}(-1)=\{b_2 \in \boldsymbol{Z}: A \cap B=\phi\}$ and $\boldsymbol{v}_j=\{b_2 \in \boldsymbol{Z}: A_j \cap B \neq \phi\}$ $(0 \leqq j \leqq u_1-1)$. Then obviously $\boldsymbol{b}(-1)=\boldsymbol{Z}-\bigcup_{j=0}^{u_1-1} \boldsymbol{v}_j$.

Lemma 1. *We take* $t \in \boldsymbol{N}$ *such that* $u_1t \equiv u_2 \pmod q$. *Then we have* $\rho(\boldsymbol{v}_0)=\rho([-a_2, u_2(a-1)-1])$ *and* $\boldsymbol{v}_j=T_{jt}\langle \boldsymbol{v}_0\rangle$ $(1 \leqq j \leqq u_1-1)$.

Lemma 1 implies that $\sigma(\boldsymbol{v}_j)$ is a segment of $C(q)$ starting from $P(j)=\sigma(-a_2+jt)$. Since $(u_1, u_2)=1$, we can take two integers x_0 and y_0 such that $q-2u_1u_2(a-1)=x_0u_1+y_0u_2$ and $1 \leqq y_0 \leqq u_1$.

Lemma 2. *Put* $J_1=[0, y_0-1]$ *and* $J_2=[y_0, u_1-1]$. *Then the following statements hold*:

(i) *If* $j \in J_1$, *then* $\sigma(x_0+u_2(2a-1))P(j)=P(j+u_1-y_0)$.

(ii) *If* $j \in J_2$, *then* $\sigma(x_0+2u_2(a-1))P(j)=P(j-y_0)$.

Lemma 3. *If* $x_0>u_2$, *then* $\sigma(\boldsymbol{b}(-1))$ *is composed of* y_0 *segments of* $C(q)$ *with the equal length* x_0, *and of* u_1-y_0 *segments of* $C(q)$ *with the equal length* x_0-u_2. *If* $1 \leqq x_0 \leqq u_2$, *then* $\sigma(\boldsymbol{b}(-1))$ *is composed of* y_0 *segments of* $C(q)$ *with the equal length* x_0. *If* $x_0 \leqq 0$, *then* $\boldsymbol{b}(-1)=\phi$.

The above three lemmas lead to

Theorem 3. *Assume that* (q_i, a_i) $(i=1,2)$ *satisfy the condition of Theorem* 1. *Let the pairs* (x_0, y_0) *and* (x_1, y_1) *be defined as in the introduction, and* $t \in \boldsymbol{N}$ *as in Lemma* 1. *Then* $S(q_1, a_1, -1) \cap S(q_2, a_2, b_2)=\phi$ *holds if and only if* $\rho(b_2) \in \rho(G_1 \cup G_2)$, *where*

$$G_1=\bigcup T_{kt}\langle [a_2-u_2, a_2-u_2+x_0-1]\rangle \qquad (0 \leqq k \leqq y_0-1)$$

and

$$G_2=\bigcup T_{rt}\langle [-x_0-a_2+u_2, -a_2-1]\rangle \qquad (0 \leqq r \leqq y_1-1).$$

(If $x_0 \leqq u_2$, we take $G_2=\phi$.)

Through some calculations, we can deduce Theorems 1 and 2 from above lemmas and Theorem 3.

References

[1] A. S. Fraenkal: The bracket function and complementary set of integers. Can. J. Math., **21**, 6–27 (1969).

[2] Ivan Niven: Diophantine Approximations. Interscience, New York (1963).

[3] R. Morikawa: On eventually covering families generated by the bracket function (to appear in Bull. Liberal Arts, Nagasaki Univ. (Natural Science), **23**).