55. *On the Identification of the Intersection Form on the Middle Homology Group with the Flat Function via Period Mapping*

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§ 1. **Introduction and the statement of the result.** Let $\varphi: X \to S$ be a universal unfolding of a function with an isolated critical point (cf. (2.2)). In this situation, we introduced the concept of a primitive form $\zeta^{(0)}$, which is an element of the relative de-Rham cohomology module of the map $\varphi: X \to S$, satisfying a certain system of bilinear differential equations on S (cf. [3] (3.2)).

Using the primitive form $\zeta^{(0)}$ (which automatically determines an infinite sequence $\zeta^{(k)}$, $k \in \boldsymbol{Z}$ of de-Rham cohomology classes), one defines a period mapping. (For simplicity, in this note we assume that n=dimension of the fiber $X_t = \varphi^{-1}(t)$, $t \in S$ of φ is even.) Namely it is given as (cf. (2.4) v));

$$(1.1)\quad P: \tilde{S} \to H^n(X_t, \boldsymbol{C}),\ \tilde{s} \in \tilde{S} \mapsto \left\{ \gamma \in H_n(X_t, \boldsymbol{Z}) \mapsto \int_{\gamma(\tilde{s})} \zeta^{(n/2-1)} \in \boldsymbol{C} \right\},$$

where $\tilde{S}$ is the monodromy covering of $S-D$ (D is the discriminant divisor in S of the map φ) and $H^n(X_t, \boldsymbol{C})$ is the middle cohomology group of a general fiber X_t of φ.

We have also introduced the concept of a flat function z on S associated with the primitive form $\zeta^{(0)}$ by

$$(1.2)\quad dz = \sum_{i=1}^{n} K^{(0)}(\nabla_{\partial/\partial t_i} \zeta^{(-1)}, \zeta^{(0)}) dt_i, \qquad Ez = (1-s)z$$

(cf. (2.4) iv)).

Then, in this note, we prove the following

Theorem 1. *Assume that the Poincaré duality σ on the middle homology of the general fiber X_t of φ*

$$(1.3)\quad \sigma: H_n(X_t, \boldsymbol{Z}) \to H^n(X_t, \boldsymbol{Z})$$

is non-degenerate. Or, equivalently, that the intersection pairing

$$(1.4)\quad I: H_n(X_t, \boldsymbol{Z}) \times H_n(X_t, \boldsymbol{Z}), \qquad (\gamma, \gamma') \mapsto \langle \sigma(\gamma), \gamma' \rangle$$

is non-degenerate.

Then there exist constant numbers c, s such that the following diagram is commutative:

$$\begin{tikzcd} \tilde{S} \arrow[rr, "P"] \arrow[dr, "2(1-s)^{-1}cz"'] & & H^n(X_t, \boldsymbol{C}) \arrow[dl, "Q"] \\ & \boldsymbol{C} & \end{tikzcd}$$

where Q is the quadratic form on $H^n(X_t, \boldsymbol{C})$ defined by

(1.5) $Q: e \in H^n(X_t, \boldsymbol{C}) \mapsto \langle \sigma^{-1}e, e \rangle \in \boldsymbol{C}$.

For the proof in § 2 we need to recall some basic concepts and results about primitive forms for a universal unfolding of a hypersurface, which are developed in [2], [3]. The proof of Theorem 1, given in § 3, is then a straightforward consequence of the algebraic representation formula for the intersection form (cf. (2.4) vi)).

§ 2. Primitive forms for a universal unfolding of a function.

We recall several concepts and construction from [2], [3]. More details are found in the references.

(2.1) Let
$$\begin{array}{ccc} (Z,0) & \xrightarrow{\hat{\pi}} & (X,0) \\ \downarrow p & & \downarrow q \\ (S,0) & \xrightarrow[\pi]{} & (T,0) \end{array}$$
be a Cartesian diagram between smooth varieties with base points 0. Assume that p, q are submersions of relative dimension $n+1$ and $\hat{\pi}, \pi$ are submersions of relative dimension 1. Assume further that there are vector fields $\hat{\delta}_1$ and δ_1 on Z and S respectively such that $p_*\hat{\delta}_1 = \delta_1$ and $\hat{\pi}^{-1}\mathcal{O}_X = \{g \in \mathcal{O}_Z : \hat{\delta}_1 g = 0\}$, $\pi^{-1}\mathcal{O}_T = \{g \in \mathcal{O}_S : \delta_1 g = 0\}$.

For convenience we employ local coordinates at 0. Namely, let $t' = (t_2, \cdots, t_m)$ be a local coordinate system for $(T, 0)$ and $t = (t_1, t')$ be a local coordinate system for $(S, 0)$ such that $\delta_1 t_1 = 1$, and $(x, t') = (x_0, \cdots, x_n, t_2, \cdots, t_m)$ are local coordinates for $(X, 0)$. Hence $(x, t) = (x, t_1, t')$ are local coordinates for $(Z, 0)$, and $\hat{\delta}_1$ and δ_1 are described by $\partial/\partial t_1$ in terms of these coordinates.

(2.2) Definition. A function $F(x, t)$ on Z is a universal unfolding of a function $f(x) := F(x, 0)$ if it satisfies i) $\partial F/\partial t_1 = 1$ ii) $\partial F/\partial x_0, \cdots, \partial F/\partial x_n$ form a parameter system in $\mathcal{O}_{Z,0}$ iii) $\partial F/\partial t_1, \cdots, \partial F/\partial t_m$ form $\mathcal{O}_{T,0}$ free basis of $\mathcal{O}_{Z,0}/(\partial F/\partial x_0, \cdots, \partial F/\partial x_n)\mathcal{O}_{Z,0}$.

If $F(x, t)$ is given, let us denote by φ the composition of the map $\hat{\pi}|^{-1}_{\{F(x,t)=0\}} : X \cong \{F(x, t) = 0\}$ with $p|_{\{F(x,t)=0\}} : \{F(x, t) = 0\} \to S$. We shall often not make the distinction between the map $\varphi : (X, 0) \to (S, 0)$ and the universal unfolding $F(x, t)$.

(2.3) Denote $\mathcal{G} := \sum_{i=1}^{m} \mathcal{O}_T \dfrac{\partial}{\partial t_i} = \{\delta \in \pi_* \mathrm{Der}_S : [\delta_1, \delta] = 0\}$.

Definition. An element $\zeta^{(0)} \in \Gamma(S, \mathcal{H}_F^{(0)})$, $\mathcal{H}_F^{(0)} := \varphi_* \Omega_X^{n+1}/dF_1 \wedge d\varphi_* \Omega_X^{n-1} + \Omega_T^1 \wedge \varphi_* \Omega_X^n$ is called a primitive form if

i) $\nabla_E \zeta^{(0)} = (r-1)\zeta^{(0)}$

ii) $K^{(k)}(\nabla_\delta \zeta^{(-1)}, \nabla_{\delta'} \zeta^{(-1)}) = 0$ for $k \geq 1$, $\delta, \delta' \in \mathcal{G}$

iii) $K^{(k)}(\nabla_\delta \nabla_{\delta'} \zeta^{(-2)}, \nabla_{\delta''} \zeta^{(-1)}) = 0$ for $k \geq 2$, $\delta, \delta', \delta'' \in \mathcal{G}$

iv) $K^{(k)}(t_1 \nabla_\delta \zeta^{(-1)}, \nabla_{\delta'} \zeta^{(-1)}) = 0$ for $k \geq 2$, $\delta, \delta' \in \mathcal{G}$.

Here, ∇ is the covariant differentiation by the Gauß-Manin connection,

E is the Euler vector field on S defined by $t_1\delta_1 - t_1*\delta_1$ (where $t_1*\delta_1$ is the element of $\mathcal{G}$ s.t. $(t_1*\delta_1)F \equiv t_1 \bmod (\partial F/\partial x_0, \cdots, \partial F/\partial x_n)$), r is the smallest exponent, $K^{(k)}$, $k \in \boldsymbol{Z}$ are higher residue pairings defined on $\pi_*\mathcal{H}_F^{(0)}$ (cf. [1], [4]) and $\zeta^{(k)} := (\nabla_{\delta_1})^k \zeta^{(0)}$.

(2.4) i) $\zeta^{(0)}$ induces a non-degenerate $\mathcal{O}_T$-bilinear form,

$$J : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{O}_T, \qquad (\delta, \delta') \longmapsto K^{(0)}(\nabla_\delta \zeta^{(-1)}, \nabla_{\delta'} \zeta^{(-1)}).$$

ii) $\zeta^{(0)}$ induces an $\mathcal{O}_T$-endomorphism, $N : \mathcal{G} \to \mathcal{G}$, by $J(N\delta, \delta')$; $= K^{(1)}(t_1 \nabla_\delta \zeta^{(-1)}, \nabla_{\delta'} \zeta^{(-1)})$. In particular $N\delta_1 = r\delta_1$. The eigenvalues of N are called the exponents.

iii) $\zeta^{(0)}$ induces a torsion-free integrable connection $\nabla/ : \mathrm{Der}_T \times \mathcal{G} \to \mathcal{G}$, by $J(\nabla/_\delta \delta', \delta'') := K^{(1)}(\nabla_\delta \nabla_{\delta'} \zeta^{(-2)}, \nabla_{\delta''} \zeta^{(-1)})$. A coordinate system $(t_1, \cdots, t_m)$ is called, flat, if $\nabla/(\partial/\partial t_i) = 0$, $i = 1, \cdots, m$.

iv) $\zeta^{(0)}$ induces a flat function z on S by the relations $dz := \sum_{i=1}^m K^{(0)}(\nabla_{\partial/\partial t_i} \zeta^{(-1)}, \zeta^{(0)}) dt_i$, $Ez = (1-s)z$, where $s = n+1-2r =$ maximal exponent-smallest exponent.

v) $\zeta^{(0)}$ induces a period mapping,

$$P : \tilde{S} \longrightarrow H^n(X_t, \boldsymbol{C}), \qquad \tilde{s} \longmapsto \left\{ \gamma \in H_n(X_t, \boldsymbol{Z}) \longmapsto \int_{\gamma(\tilde{s})} \zeta^{(n/2-1)} \in \boldsymbol{C} \right\}$$

where $\tilde{S}$ is the monodromy covering of the fibration $X \to S$, and t is a generic point of $\tilde{S}$. Here $\gamma(\tilde{s})$ is the image of $\gamma \in H_n(X_t, \boldsymbol{Z})$ in $H_n(X_{\tilde{s}}, \boldsymbol{Z})$ by the parallel translation for any $\tilde{s} \in \tilde{S}$.

By definition v), the period map P is of maximal rank, iff there exist no integral exponents.

vi) The intersection number $I(\gamma, \gamma')$ of (1.4) is expressed as follows;

$$I(\gamma, \gamma') = c^{-1} \sum_{i=1}^m \left(N - \frac{n}{2}\right) \frac{\partial}{\partial t_i} \int_{\gamma(\tilde{s})} \zeta^{(n/2-2)} \left(\frac{\partial}{\partial t_i}\right)^* \int_{\gamma'(\tilde{s})} \zeta^{(n/2-1)}$$

where c is a constant and $(\partial/\partial t_i)^*$, $i = 1, \cdots, m$ is the dual basis of $\mathcal{G}$ with respect to J of (2.4) i).

It follows directly from this expression that the pairing I is non-degenerate iff there exist no integral exponents.

§3. **A proof of Theorem 1.** (3.1) *Let* $A : \tilde{S} \to H_n(X_t, \boldsymbol{C})$ be the composition $\sigma^{-1}P$ of (1.1) and (1.3).

Using $\boldsymbol{Z}$-basis $\gamma_1, \cdots, \gamma_m$ of $H_n(X_t, \boldsymbol{Z})$, define $A^i(\tilde{s})$ by,

$$A(\tilde{s}) = \sum_{i=1}^m A^i(\tilde{s}) \gamma_i \qquad \text{for } \tilde{s} \in \tilde{S}.$$

(3.2) From the definitions of the pairing I of (1.4) and the map A, one gets

$$I(A(\tilde{s}), \gamma) = \langle P(\tilde{s}), \gamma \rangle = \int_{\gamma(\tilde{s})} \zeta^{(n/2-1)} \qquad \text{for } \tilde{s} \in \tilde{S}.$$

(3.3) Let $t_1, \cdots, t_m$ be a flat coordinate system such that $\delta_1 = \partial/\partial t_1$. Let $t_1^*, \cdots, t_m^*$ be the dual coordinate system w.r.t. J of (2.4) i). Then we have $dz=dt_1^*$. ($\because dz=\sum_{i=1}^m K^{(0)}(\nabla_{\partial/\partial t_i^*}\zeta^{(-1)}, \zeta^{(0)})dt_i^*=\sum_{i=1}^m J(\partial/\partial t_i^*, \partial/\partial t_1)dt_i^* =dt_1^*$).

(3.4) Now in the formula (2.4) vii), substitute γ by $A(\tilde{s}) =\sum_{i=1}^m A^i(\tilde{s})\gamma_i$ and γ' by γ_k, $k=1, \cdots, m$ so that one obtains;

i) $$c\int_{\gamma_k(\tilde{s})}\zeta^{(n/2-1)}=cI(A(\tilde{s}), \gamma_k)$$
$$=\sum_{i,j=1}^m A^j(\tilde{s})\left(N-\frac{n}{2}\right)\frac{\partial}{\partial t_i}\int_{\gamma_j(\tilde{s})}\zeta^{(n/2-2)}\frac{\partial}{\partial t_i^*}\int_{\gamma_k(\tilde{s})}\zeta^{(n/2-1)}.$$

By assumption on σ, there exist no integral exponents, and therefore the period map P is of maximal rank. Hence $\langle\gamma_k, P(\tilde{s})\rangle=\int_{\gamma_k(\tilde{s})}\zeta^{(n/2-1)}$ $k=1, \cdots, m$ may be regarded as coordinates for $\tilde{s}$. Thus multiplying the above by the inverse matrix of $(\partial/\partial t_i^*\int_{\gamma_k(\tilde{s})}\zeta^{(n/2-1)})_{i,k=1,\cdots,m}$, one gets,

ii) $$c\left(r-\frac{n}{2}\right)^{-1}Et_i^*=\sum_{j=1}^m A^j(\tilde{s})\left(N-\frac{n}{2}\right)\frac{\partial}{\partial t_i}\int_{\gamma_j(\tilde{s})}\zeta^{(n/2-2)}.$$

$\Big($Note that $E=\left(r-\frac{n}{2}\right)\sum_{k=1}^m \gamma_k\frac{\partial}{\partial\gamma_k}$, since $E\int_{\gamma_k(\tilde{s})}\zeta^{(n/2-1)}$
$$=\left(r-\frac{n}{2}\right)\int_{\gamma_k(\tilde{s})}\zeta^{(n/2-1)} \text{ for } k=1, \cdots, m.\Big)$$

In the formula ii) we put $i=1$. Noting (2.4) ii) iv) and (3.3), we get the last formula,

iii) $$c\left(r-\frac{n}{2}\right)^{-1}z=\sum_{j=1}^m A^j(\tilde{s})\int_{\gamma_j(\tilde{s})}\zeta^{(n/2-1)}=I(A(\tilde{s}), A(\tilde{s})) \quad (\because (3.2)).$$

This completes the proof of Theorem 1.

References

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