48. On the Convergence of $\sum_{n=1}^{\infty} n^{-\alpha}\sin(n^{\beta}\theta)$

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0. Convergence problem of $\sum_{n=1}^{\infty} n^{-\alpha}\sin(n^{\beta}\theta)$ is decided when $0<\alpha\leqq 1$, $0\leqq\beta\leqq 1$ (cf. [3] Theorem 84), but not when $0<\alpha\leqq 1$, $1<\beta$.

From a known result relating to the Gaussian sum (cf. [2] Theorem 4.15, also [4]), we have for $\theta=2l\pi/(4m+1)$, $(2l+1)\pi/(2m+1)$

$$\sum_{k=1}^{n}\sin(k^2\theta)=O(1), \tag{1}$$

and for $\theta=(2l+1)\pi/2m$, $2l\pi/(4m+3)$

$$\sum_{k=1}^{n}\sin(k^2\theta)=Bn+O(1). \tag{2}$$

Hence for example, by partial summation

$$\sum_{n=1}^{\infty} n^{-\alpha}\sin(n^2\theta) \tag{3}$$

converges for $\theta=2l\pi/(4m+1)$*) and diverges for $\theta=(2l+1)\pi/2m$, provided $0<\alpha\leqq 1$.

On the one hand, Wilton ([9] Theorem B, cf. also [8]) showed among other things that when $0<\alpha<1$, $1<\beta\leqq 2-2\alpha$

$$\sum_{n=1}^{\infty} n^{-\alpha}\exp(in^{\beta}\theta) \qquad (i^2=-1) \tag{4}$$

diverges for all $\theta>0$.

In this paper we prove the following

Theorem. *If $\alpha>0$ and $1<\beta<2\alpha$, then* (4) *converges for all $\theta>0$.***)

1. *Proof of Theorem.* By the Euler summation formula,

$$\begin{aligned}\sum_{m=1}^{n}\frac{\sin(m^{\beta}\theta)}{m^{\alpha}}&=\frac{1}{2}\left(\sin\theta+\frac{\sin(n^{\beta}\theta)}{n^{\alpha}}\right)+\int_1^n\frac{\sin(t^{\beta}\theta)}{t^{\alpha}}\,dt\\&\quad+\beta\theta\int_1^n\frac{\phi(t)\cos(t^{\beta}\theta)}{t^{\alpha-\beta+1}}\,dt-\alpha\int_1^n\frac{\phi(t)\sin(t^{\beta}\theta)}{t^{\alpha+1}}\,dt\\&=\frac{1}{2}\left(\sin\theta+\frac{\sin(n^{\beta}\theta)}{n^{\alpha}}\right)+I_1^n+\beta\theta I_2^n-\alpha I_3^n,\ \text{say,}\end{aligned}$$

where

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*) [6] appears to contain some incorrect arguments. It also contradicts to [9] e.g. when $0<\alpha\leqq 1/3$.

**) Note that we cannot admit $\beta=2\alpha$ for $\alpha<1/2$ in Wilton's, and for $\alpha>1/2$ in our theorem as shown by (2), (3).

In another way, we have proved that (4) converges for all $\theta>0$, provided $\alpha=1/2$, $\beta<3/2$ (cf. [1]).

$$\phi(t)=-\sum_{k=1}^{\infty}\frac{\sin(2k\pi t)}{k\pi}.$$

It is easy to see that I_1^n and I_3^n converge as $n\to\infty$.

By termwise integration,

$$I_2^n=-\sum_{k=1}^{\infty}\frac{1}{k\pi}\int_1^n\frac{\sin(2k\pi t)\cos(t^{\beta}\theta)}{t^{\alpha-\beta+1}}dt.$$

Then we can prove

$$\sum_{k=1}^{\infty}\int_1^{\infty}\frac{\sin(2k\pi t)\cos(t^{\beta}\theta)}{k\pi t^{\alpha-\beta+1}}dt=\int_1^{\infty}\sum_{k=1}^{\infty}\frac{\sin(2k\pi t)\cos(t^{\beta}\theta)}{k\pi t^{\alpha-\beta+1}}dt, \tag{5}$$

provided the left hand side exists (cf. e.g. [5] § 218).

To prove that the left hand side of (5) exists, we write

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{1}{k\pi}\int_1^n\frac{\sin(2k\pi t)\cos(t^{\beta}\theta)}{t^{\alpha-\beta+1}}dt\\&=\sum_{k=1}^{\infty}\frac{1}{2k\pi}\left(\int_1^n\frac{\sin(t^{\beta}\theta+2k\pi t)}{t^{\alpha-\beta+1}}dt-\int_1^n\frac{\sin(t^{\beta}\theta-2k\pi t)}{t^{\alpha-\beta+1}}dt\right)\\&=\sum_{k=1}^{\infty}\frac{1}{2k\pi}(G_k^n-G_{-k}^n),\quad\text{say.}\end{aligned}$$

Since $\lim_{n\to\infty}G_k^n$ converges and is $O(1/k)$ (van der Corput, cf. e.g. [7] Lemma 4.3), we consider $\lim_{n\to\infty}G_{-k}^n$ below. Put $F(t)=t^{\beta}\theta-2k\pi t$ and $G(t)=1/t^{\alpha-\beta+1}$. Then $F'(t)=\beta\theta t^{\beta-1}-2k\pi$ and $F'(t)=0$ for $t_0=(2k\pi/\beta\theta)^{1/(\beta-1)}$. For any fixed positive integer k, $F'(t)>0$ and $G(t)/F'(t)$ is monotonic when $t>t_0$. Also for $t\geqq c>t_0$

$$F'(t)/G(t)\geqq\beta\theta c^{\alpha}-2k\pi c^{\alpha-\beta+1}>0.$$

Therefore by van der Corput's lemma,

$$\int_1^{\infty}\frac{\sin(t^{\beta}\theta-2k\pi t)}{t^{\alpha-\beta+1}}dt$$

exists. Then

$$\begin{aligned}G_{-k}^n&=\int_1^n\frac{(2k\pi-\beta\theta t^{\beta-1})\sin(t^{\beta}\theta-2k\pi t)}{2k\pi t^{\alpha-\beta+1}}dt+\int_1^n\frac{\beta\theta t^{\beta-1}\sin(t^{\beta}\theta-2k\pi t)}{2k\pi t^{\alpha-\beta+1}}dt\\&=\frac{1}{2k\pi}\left(\frac{\cos(n^{\beta}\theta)}{n^{\alpha-\beta+1}}-\cos\theta+(\alpha-\beta+1)\int_1^n\frac{\cos(t^{\beta}\theta-2k\pi t)}{t^{\alpha-\beta+2}}dt\right.\\&\qquad\left.+\beta\theta\int_1^n\frac{\sin(t^{\beta}\theta-2k\pi t)}{t^{\alpha-2\beta+2}}dt\right).\end{aligned}$$

Now we put

$$\begin{aligned}\int_1^{\infty}\frac{\sin(t^{\beta}\theta-2k\pi t)}{t^{\alpha-2\beta+2}}dt&=\int_1^{t_0/2}+\int_{t_0/2}^{t_0-\delta}+\int_{t_0-\delta}^{t_0+\delta}+\int_{t_0+\delta}^{\infty}\\&=J_1+J_2+J_3+J_4,\quad\text{say,}\end{aligned}$$

where $(t_0>)\delta>0$ is exactly determined later.

Put $f(t)=t^{\beta-1}$. Using the second mean value theorem twice,

$$J_1=\int_1^{t_0/2}\left(\frac{f(t)}{-F'(t)}\right)G(t)(-F'(t)\sin F(t))dt$$

$$=\frac{f(t_0/2)}{-F'(t_0/2)}G(t_1)\int_{t_1}^{t_2}(\cos F(t))'dt$$

$$=O\left(\frac{1}{\beta\theta(1-1/2^{\beta-1})}\right).^{***)}$$

Similarly $J_2=O(1/\delta\beta(\beta-1)\theta t_0^{\alpha-\beta})$,

$J_3=O(\delta/t_0^{\alpha-2\beta+2})$,

$J_4=O(1/\delta t_0^{\alpha-\beta})$.

Now we take $\delta=t_0^{1-\beta/2}(<t_0)$. Then

$$J_1+J_2+J_3+J_4=O(1)+O(1/t_0^{\alpha-(3/2)\beta+1}).$$

Therefore if $\beta<2\alpha$,

$$\sum_{k=1}^{\infty}\frac{1}{(2k\pi)^2}\int_1^{\infty}\frac{\sin(t^{\beta}\theta-2k\pi t)}{t^{\alpha-2\beta+2}}dt$$

$$=O\left(\sum_{k=1}^{\infty}1/k^2\right)+O\left(\sum_{k=1}^{\infty}1/k^2t_0^{\alpha-(3/2)\beta+1}\right)$$

$$=O(1)+O\left(\sum_{k=1}^{\infty}1/k^{2+(\alpha-(3/2)\beta+1)/(\beta-1)}\right)$$

$$=O(1)+O\left(\sum_{k=1}^{\infty}1/k^{1+(\alpha-(\beta/2))/(\beta-1)}\right)=O(1).$$

Hence the proof is completed.

References

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[9] ——: ditto. II. ibid., **9**, 247–254 (1934).

***) The constants implied by the O's for J_1–J_4 are all absolute.