47. On the Stickelberger Ideal and the Relative Class Number

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(Communicated by Shokichi IYANAGA, M. J. A., April 12, 1982)

A finite abelian extension of $\boldsymbol{Q}$ contained in $\boldsymbol{C}$ will be called an abelian field. Let k be an imaginary abelian field, namely an abelian field not contained in $\boldsymbol{R}$. We denote by $R=\boldsymbol{Z}[G]$ the group ring of the Galois group $G=\mathrm{Gal}\,(k/\boldsymbol{Q})$ over $\boldsymbol{Z}$. Put

$$A=\left\{\alpha\in R\,;(1+J)\alpha=a\sum_{\sigma\in G}\sigma \text{ for some } a\in \boldsymbol{Z}\right\}$$

where J denotes the complex conjugation of k. Let Q be the unit index of k, g_k the number of distinct rational primes ramifying in k, and c_k the rational number which describes the difference between the relative class number h_k^- of k and the group index $[A:S]$ where S denotes the Stickelberger ideal of k (for the definition of a Stickelberger ideal, see [4]):

$$c_k h_k^- = [A:S].$$

It is known that $d=Qc_k$ is a natural number. In the case $g_k=1$ or 2, W. Sinnott has determined the number d ([4]). In this article, we give some results concerning the range of c_k.

Theorem. *In general, $2c_k$ is a natural number, and the following assertions hold.*

1) *If $g_k=1$, then we have $c_k=1$.*

2) *If $g_k=2$, then we have $c_k=1/2$ or 1, and there exist infinitely many imaginary abelian fields k for each case.*

3) *If $g_k=3$, then we have $c_k=2^a$ for some integer $a\geqq -1$. On the other hand, for any given integer $a\geqq -1$, there exist infinitely many imaginary abelian fields k with $g_k=3$ and $c_k=2^a$.*

4) *For any given pair (m,n) of natural numbers with $m\geqq 4$, there exist infinitely many imaginary abelian fields k satisfying $g_k=m$ and $c_k=n/2$.*

Remark. The assertion 1) is obtained immediately from Proposition 5.2 in [4] and Satz 23 in [1].

Using Theorem and W. Sinnott's result (Theorem in [3]), we obtain the following

Proposition. *For any given pair (m,n) of natural numbers*

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satisfying $m \geqq 4$, *there exist infinitely many cyclotomic fields* k *with* $g_k = m$ *and* $n \mid h_k^-$.

Remark. The assertion of Proposition also holds with the additional condition that k is tamely ramified over $\boldsymbol{Q}$.

The details will appear elsewhere.

References

[1] H. Hasse: Über die Klassenzahl abelscher Zahlkörper. Akademie-Verlag, Berlin (1952).
[2] K. Iwasawa: A class number formula for cyclotomic fields. Ann. of Math., **76**, 171–179 (1962).
[3] W. Sinnott: On the Stickelberger ideal and the circular units of a cyclotomic field. ibid., **108**, 107–134 (1978).
[4] ——: On the Stickelberger ideal and the circular units of an abelian field. Invent. math., **62**, 181–234 (1980).