40. A Perturbation Theory for Abstract Evolution Equations of Second Order

By Michiaki WATANABE
Faculty of General Education, Niigata Univeristy

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1. **Introduction.** Let X be a Banach space with norm $\|\cdot\|$ and $B(Y,X)$ be the totality of bounded linear operators mapping Y into X. A subset $C(t)$, $t\in R$ of $B(X,X)$ is called a strongly continuous cosine family in X if

(1) $C(t+s)+C(t-s)=2C(t)C(s)$ for all $t,s\in R$;

(2) $C(0)=I$;

(3) $C(t)x$ is continuous in $t\in R$ for each fixed $x\in X$.

The associated sine family is given by

$$S(t)x=\int_0^t C(r)xdr$$

for $x\in X$ and $t\in R$. The infinitesimal generator is the operator $A:D(A)\to X$ defined by $Ax=\lim_{h\to 0}2h^{-2}(C(h)-I)x$ for $x\in D(A)$, where $D(A)=\{x\in X:\lim_{h\to 0}h^{-2}(C(h)-I)x \text{ exists}\}$. It is well known that for $\lambda>\omega$, λ^2 belongs to the resolvent set of A and for $x\in X$

$$\lambda(\lambda^2-A)^{-1}x=\int_0^\infty e^{-\lambda t}C(t)xdt, \tag{4}$$

where ω is a constant with $\omega\geqq\log_e(1+2\|C(1)\|)$. (See [1, p. 90].)

The cosine family in X with generator A is associated with the Cauchy problem for the abstract evolution equation of second order in X

$$d^2u/dt^2=Au,\quad t\in R;\quad u(0)=u,\quad u'(0)=x. \tag{5}$$

It is natural to try to convert (5) into a well-posed first order system

$$\frac{d}{dt}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}0&I\\A&0\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix},\quad t\in R;\quad \begin{pmatrix}u\\v\end{pmatrix}(0)=\begin{pmatrix}u\\x\end{pmatrix} \tag{6}$$

and to make use of the extensive theory of groups. (See, for example [6].)

For a strongly continuous cosine family $C(t)$, $t\in R$ in X with the infinitesimal generator A, we are concerned with the set

$$E=\{x\in X;\ C(t)x \text{ is once continuously differentiable in } t\in R\}.$$

Kisyński [2] proved the important facts that the set E under the norm

$$|u|_E=\|u\|+\text{Max}\,\{\|C'(s)u\|:0\leqq s\leqq 1\}$$

becomes a Banach space and that (5) can always be converted into the well-posed problem (6) in the Banach space $E\times X$.

In order to make this conversion more convenient we will, in this paper, deal with the characterization of the set E in terms of the generator A, and will then apply it to a perturbation theory for (5).

Theorem 1. *Let A be the infinitesimal generator of a strongly continuous cosine family in X. Then*

(i) *the set E is the closure of $D(A)$ with respect to the norm*

$$|u|=\|u\|+\operatorname{Sup}\{\|(1/n!)(\lambda-\omega)^{n+1}(-d/d\lambda)^n A(\lambda^2-A)^{-1}u\| : \lambda>\omega,\ n=0,1,\cdots\}$$

and satisfies, after the replacement by A of $A-b^2$, $b\in R$ if necessary,

(ii) $D((-A)^\beta)\subset E\subset D((-A)^\alpha)$ *continuously*

for any α and β with $0\leqq\alpha<1/2<\beta\leqq 1$.

Theorem 2. *Let A be the infinitesimal generator of a strongly continuous cosine family in X, and B belong to $B(E,X)$. Then $A+B$ is the infinitesimal generator of a strongly continuous cosine family in X.*

Corollary 1. *Let A be such as in the theorem. Suppose that B is a closed linear operator in X satisfying $D(B)\supset D(A)$, and that there exist positive numbers C, $\mu>\omega$ and an integer $N\geqq 0$ such that*

$$\|Bu\|\leqq C\|u\|+C\|(1/N!)(\mu-\omega)^{N+1}(-d/d\mu)^N A(\mu^2-A)^{-1}u\|$$

for all $u\in D(A)$. Then the conclusion of the theorem is true.

Corollary 2. *Let A be such as in the theorem and replace $A-b^2$, $b\in R$ if necessary. Let B belong to $B(D((-A)^\alpha), X)$, $0\leqq\alpha<1/2$, where $D((-A)^\alpha)$ is the domain of $(-A)^\alpha$ with the graph norm. Then the conclusion of the theorem is true.*

Corollary 3 (Travis and Webb [5]). *Let A be such as in the theorem. Let B be a closed linear operator in X such that $S(t)X\subset D(B)$ for all $t\in R$ and $BS(t)x$ is continuous in $t\in R$ for each fixed $x\in X$. Then the conclusion of the theorem is true.*

2. Characterization of E. We begin with recalling some properties of a strongly continuous cosine family $C(t)$, $t\in R$ in X with the infinitesimal generator A, of which we will make later use.

(7) $$S(t+s)-S(t-s)=2C(t)S(s) \qquad \text{for } t,s\in R,$$

(8) $$C(t+s)-C(t-s)=2AS(t)S(s) \qquad \text{for } t,s\in R,$$

(9) $$\lim_{h\to 0}\frac{2}{h^2}\int_0^h S(r)x\,dr=x \qquad \text{for } x\in X,$$

(10) $$(C(t)-I)x=A\int_0^t S(r)x\,dr \qquad \text{for } x\in X \text{ and } t\in R.$$

Since A is a closed linear operator in X, (10) is a simple consequence of (9) and the equality

$$(C(t)-I)\int_0^h S(r)x\,dr=(C(h)-I)\int_0^t S(r)x\,dr.$$

Lemma 1. *The following statements are mutually equivalent:*

(i) $x\in E$;

(ii) $S(t)x\in D(A)$ *for $t\in R$ and $AS(t)x$ is continuous in $t\in R$;*

(iii) *$S(t)x \in D(A)$ for $t \in [0,1]$ and $AS(t)x$ is continuous in $t \in [0,1]$.*

In this case $C'(t)x = AS(t)x$, and for $\omega \geqq \log_e(1+2\|C(1)\|)$

$$\|C'(t)x\| \leqq e^{\omega|t|} \operatorname{Max}\{\|C'(s)x\| : 0 \leqq s \leqq 1\}. \tag{11}$$

Proof. The equivalence will be clear from (7), (9) and (10). Making use of (7) and choosing ω so large that $e^{-\omega}+2\|C(1)\|e^{-\omega} \leqq 1$, we can prove by induction on n that (11) is valid for $t \in [0,n]$, $n=1,2,\cdots$. Q.E.D.

Lemma 2. *Replace $A-b^2$, $b \in R$ by A if necessary. Then E is included in $D((-A)^\alpha)$, $0 \leqq \alpha < 1/2$ and for all $u \in E$*

$$\|(-A)^\alpha u\| \leqq C_\alpha |u|_E \tag{12}$$

with some constant $C_\alpha > 0$ independent of u (cf. [4, Proposition]).

Proof. Integration of (4) by parts yields

$$A(\lambda^2-A)^{-1}u = \int_0^\infty e^{-\lambda t} C'(t)u\,dt \qquad \text{for } u \in E \text{ and } \lambda > \omega, \tag{13}$$

which with (11) implies $\|A(\lambda^2-A)^{-1}u\| \leqq (\lambda-\omega)^{-1}(|u|_E - \|u\|)$.

Using [7, Theorem], we have that u belongs to $D((-A)^\alpha)$ and

$$(-A)^\alpha u = \frac{2\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{2\alpha-1}(-A)(\lambda^2-A)^{-1}u\,d\lambda. \tag{14}$$

Q.E.D.

The proof of Theorem 1. Differentiating (13), we have

$$(-d/d\lambda)^n A(\lambda^2-A)^{-1}u = \int_0^\infty e^{-\lambda t} t^n C'(t)u\,dt \qquad \text{for } u \in E,$$

which together with (11) implies that $|u| \leqq |u|_E$ for $u \in E$. Since $D(A)$ is a dense subset of E, E is included in the closure F of $D(A)$ with respect to the norm $|\cdot|$.

Put $S_n(t) = (1/n!)\lambda^{n+1}(-d/d\lambda)^n(\lambda^2-A)^{-1}|_{\lambda=n/t}$ for $t>0$ with $S_n(0)=0$. Then for $u \in D(A)$, $S_n(t)Au$ is dominated in $\|\cdot\|$ by $(1-t\omega/n)^{-n-1}(|u|-\|u\|)$ and tends to $AS(t)u$ as $n \to \infty$ by the Post-Widder theorem. Thus we obtain $|u|_E - \|u\| \leqq e^\omega(|u|-\|u\|)$ for $u \in D(A)$. Using this inequality, we can prove $F \subset E$ and complete the proof of (i).

Lemma 2 asserts that the latter half of (ii) is true.

Making use of (8) and (12), we have for $x \in X$ and $t \in [0,1]$

$$\|(-A)^{1-\beta}S(t)x\| \leqq C_{1-\beta}|S(t)x|_E \leqq D_\beta \|x\|.$$

The continuity in $t \in [0,1]$ of $(-A)^{1-\beta}S(t)x$ is clear from (14) with α and u replaced by $1-\beta$ and $S(t)x$ respectively. Therefore, if u belongs to $D((-A)^\beta)$, then $-AS(t)u$ equals to $(-A)^{1-\beta}S(t)\cdot(-A)^\beta u$, and is continuous in $t \in [0,1]$. Thus we obtain by Lemma 1 that u belongs to E and $|u|_E - \|u\| \leqq D_\beta \|(-A)^\beta u\|$. Q.E.D.

3. Proof of perturbation results. *The proof of Theorem* 2. The operator $\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ with domain $D(A)\times E$ generates a strongly continuous group in $E \times X$ by Kisyński's theorem [2]. So does $\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$ with domain $D(A)\times E$ by [3, Theorem 3.4] since $\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$ is in $B(E \times X$,

$E\times X$). Thus, using again Kisyński's theorem, we obtain the desired result for $A+B$.

Corollaries 1 and 2 are easy consequences of Theorem 1.

The proof of Corollary 3. If $u\in E$, then by (8)

$$u=\frac{1}{2}\int_0^2 C(t)udt-\frac{1}{2}\int_0^2 (C(t)-I)udt$$
$$=\frac{1}{2}S(2)u-\int_0^2 S(t/2)AS(t/2)udt.$$

Hence the assumption implies that u belongs to $D(B)$ and

$$Bu=\frac{1}{2}BS(2)u-\int_0^2 BS(t/2)\cdot AS(t/2)udt,$$

proving that B belongs to $B(E, X)$. Q.E.D.

Remarks. Theorem 1, (i) suggests that the Hille-Yosida-Phillips theorem for $\begin{pmatrix}0 & I\\ A & 0\end{pmatrix}: D(A)\times F\to F\times X$ is used to obtain a new proof of the Da Prato-Giusti-Fattorini-Sova theorem on the generation of cosine families in X.

The algebraical inclusion of the latter half of Theorem 1, (ii) was proved by Rankin, III [4]. The method is, however, quite different from ours.

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